Engineering Numerical Methods

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1-Introduction

Numerical methods

Numerical methods are a class of techniques used for solving a wide variety of mathematical problems in terms of numbers using only arithmetic and logic operations. The main advantage of numerical methods is their ability to solve problems that can not be treated using classical analytical mathematics, such as non-linearity and complex geometries. The disadvantage is that the solutions using numerical methods are iterative, approximate, and not exact as those obtained by analytical methods.

Errors in numerical computations

1- Errors from the method of solution, since all numerical methods are only approximate.

2- Errors from solution truncation, since numerical methods are iterative and the iterations can not continue infinite times.

3- Errors from numbers round off.

4- Errors from the mathematical model of the physical problems. For example in flexural formula the dx^2 term is neglected , also usually $\sin\theta$ is approximated to θ for small values of the later.

Error calculation

If x_{approx} is an approximate to x_{exact} then:

1- The absolute error is E or $\Delta = \left| x_{exact} - x_{approx} \right|$.

2- The relative error is $R = \left| \dfrac{x_{exact} - x_{approx}}{x_{exact}} \right|$.

3- The percent relative error is $P = \left| \dfrac{x_{exact} - x_{approx}}{x_{exact}} \right| \times 100$.

2- Numerical Solution of Algebraic Equations (Roots of Equations)

Introduction

A problem commonly encountered in engineering is that of determining the roots of an equation of the form $y = f(x)$. Finding the roots of an equation is equivalent to finding the values of x for which $f(x) = 0$. For this reason the roots of equation are often called the zeros of the equation. Different techniques of varying degrees of accuracy and rates of convergence were developed to determine these roots.

Root solving problem consists of finding the values of the independent variable which satisfy relationships, such as:

$$Ax^3 + Bx^2 = Cx + D.$$

The procedure for finding the roots will always be to collect all terms on one side of the equality sign, for example (for the above equation):

$$Ax^3 + Bx^2 - Cx - D = 0.$$

For any values of x other than the roots, this equation will not be satisfied. So in general:

$$f(x) = Ax^3 + Bx^2 - Cx - D.$$

Now, finding the roots of the above equation is now equivalent to finding the values of x for which $f(x)$ is zero, i.e:

$$f(x) = 0.$$

Single and multiple roots

1- x_1 is a single (simple) root if $f(x_1)=0$ and $f'(x_1)\neq 0$.

2- x_1 is a multiple root of :

multiplicity 2 if $f(x_1)=f'(x_1)=0$ and $f''(x_1)\neq 0$,

multiplicity 3 if $f(x_1)=f'(x_1)=f''(x_1)=0$ and $f'''(x_1)\neq 0$, and so on.

Accuracy in roots determination

All roots determination numerical methods are iterative, hence roots of different degrees of accuracy can be obtained depending on the method used and the number of iteration performed. The iteration should be continued until one (or more) of such following conditions are satisfied:

1- $\Delta = \left| x_i - x_{i-1} \right| \le \varepsilon$, where ε is the allowed absolute difference between two successive trials (iterations).

2- $\left| f(x_i) \right| \le E$, where E is the allowed absolute error in the value of the function.

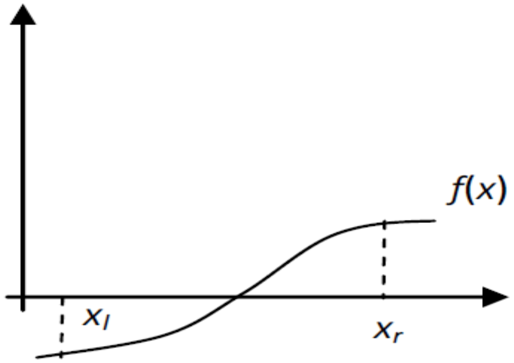
Solution of algebraic equations (determining roots of equations)

1- Bisection Method

This method, which is also known as interval halving method, is too inefficient for hand computation but is ideally suited to machine computation.

To find a real root of a given function *f(x)*, the following steps will be used:

1- Estimate two approximations of the root x_l and x_r such that $f(x_l) < 0$ ($f(x_l)$ is negative) and $f(x_r) > 0$ ($f(x_r)$ is positive).



2- Bisect the interval (x_l, x_r) to find its midpoint

$x_m = \frac{x_r + x_l}{2}$ (which is considered as an improved approximation of root).

3- Check the sign of $f(x_m)$. If $f(x_m) < 0$, this mean that the root lies between x_m and x_r, then for the next iteration let $x_l = x_m$. If $f(x_m) > 0$, this mean that the root lies between x_l and x_m, then for the next iteration let $x_r = x_m$.

4- Repeat steps 2 and 3 until the required accuracy ε is achieved.

Notes:

1- For each iteration, the root is assumed to be the midpoint of the last interval found to contain it, i.e; the root is x_m.

2- For each iteration, the maximum absolute error Δ in the value of the root is no greater than one half the size of last interval found to contain the root, i.e;

$$\Delta = \frac{|x_r - x_l|}{2} \qquad \text{or} \qquad \Delta = \left|(x_m)_i - (x_m)_{i-1}\right|.$$

3- The maximum error Δ in the value of the root in a given iteration is one half its value in the previous iteration; $\Delta_i = \frac{1}{2}\Delta_{i-1}$. So this method has very slow convergence.

4- The bisection method cannot be used to find roots of functions that do not change their sign (from positive to negative or from negative to positive).

5- Since $\Delta_i = \frac{1}{2}\Delta_{i-1}$ and $\Delta = \frac{|x_r - x_l|}{2}$, so we can estimate the number of iterations n required to find a root to an accuracy of ε as follows:

$$\Delta \le \varepsilon \;\Rightarrow\; \frac{|x_r - x_l|}{2^n} \le \varepsilon \;\Rightarrow\; 2^n.\varepsilon \ge |x_r - x_l| \;\Rightarrow\; n\ln 2 + \ln\varepsilon \ge \ln|x_r - x_l|,$$

$$\therefore n \ge \frac{\ln|x_r - x_l| - \ln\varepsilon}{\ln 2} \qquad \text{or} \qquad n \ge \frac{\ln\left|\frac{x_r - x_l}{\varepsilon}\right|}{\ln 2}.$$

Example 1: Find the root(s) of the function $f(x) = x^3 - 5x^2 - 2x + 10$ using six iterations.

Solution:

Check the sign of $f(x)$ at different values of x:

x	- 2	- 1	0	1	2	3	4	5
$f(x)$	- 14	6	10	4	- 6	- 14	- 8	0

There are three roots: The first root lies between x = -2 and x = -1, the second root lies between x = 1 and x = 2, and the third root is x = 5 (exact value).

To find the first root by using the bisection method,

<u>1st iteration:</u> Let $x_l = -2$ and $x_r = -1$.

$$x_m = \frac{x_r + x_l}{2} \quad \Rightarrow \quad x_m = \frac{-1+(-2)}{2} = -1.5$$

Check the sign of $f(x_m) \quad \Rightarrow \quad f(-1.5) = (-1.5)^3 - 5(-1.5)^2 - 2(-1.5) + 10 = -1.625$.

Since $f(x_m) < 0$, then for the next iteration $x_l = x_m = -1.5$ and $x_r = -1$ (unchanged)

<u>2nd iteration:</u> $x_l = -1.5$ and $x_r = -1$.

The calculations must be repeated as in the 1st iteration and continued until the required number of iterations is reached.

It is more preferred to put the calculations in a table as below:

No. of Iteration (i)	x_l	x_r	$x_m = \frac{x_r + x_l}{2}$	$f(x_m)$	$\Delta = \left\lvert \frac{x_r - x_l}{2} \right\rvert$
1	- 2	- 1	- 1.5	- 1.625	0.5
2	- 1.5	- 1	- 1.25	2.73….	0.25
3	- 1.5	- 1.25	- 1.375	0.69….	0.125
4	- 1.5	- 1.375	- 1.4375	- 0.42….	0.0625
5	- 1.4375	- 1.375	- 1.40625	0.14….	0.03125
6	- 1.4375	- 1.40625	- 1.421875	- 0.13….	0.015625

After six iterations the first approximate root is $x_{root} \approx -1.421875$.

To find the second root:

By using the bisection method,

<u>1^{st} iteration:</u> Let $x_l = 2$ and $x_r = 1$.

$$x_m = \frac{x_r + x_l}{2} \quad \Rightarrow \quad x_m = \frac{1+2}{2} = 1.5$$

Check the sign of $f(x_m)$ $\Rightarrow$ $f(1.5) = (1.5)^3 - 5(1.5)^2 - 2(1.5) + 10 = -0.875$.

Since $f(x_m) < 0$, then for the next iteration $x_l = x_m = 1.5$ and $x_r = 1$ (unchanged).

<u>2^{nd} iteration:</u> $x_l = 1.5$ and $x_r = 1$. And so on.

i	x_l	x_r	$x_m = \frac{x_r + x_l}{2}$	$f(x_m)$	$\Delta = \left\lvert \frac{x_r - x_l}{2} \right\rvert$
1	2	1	1.5	- 0.875	0.5
2	1.5	1	1.25	1.64….	0.25
3	1.5	1.25	1.375	0.39….	0.125
4	1.5	1.375	1.4375	- 0.42….	0.0625
5	1.4375	1.375	1.40625	0.05….	0.03125
6	1.4375	1.40625	1.421875	- 0.07….	0.015625

After six iterations the second approximate root is $x_{root} \approx 1.421875$.

<u>Example 2</u>: Find the point(s) of intersection of $y = \ln x$ and $y = 2x - 3$ accurately to three decimal places (i.e; $\varepsilon = 1\times 10^{-3}$).

Solution:

To find the point(s) of intersection we put $y_1 = y_2 \quad \Rightarrow \quad \ln x = 2x - 3$,

$$\therefore \ln x - 2x + 3 = 0 \quad \Rightarrow \quad f(x) = 0 \text{ (Root finding problem)}$$

So we must find the root(s) of $f(x)$ where $f(x) = \ln x - 2x + 3$.

Check the sign of $f(x)$ at different values of x:

x	0.01	1	2	3	4	5
$f(x)$	- 1.625	1	- 0.306	- 1.9	- 3.6	- 5.39

There are two roots: The first root lies between $x = 1$ and $x = 2$ and the second root lies between $x = 0.01$ and $x = 1$. To find the first root by using the bisection method,

<u>1^{st} iteration:</u> Let $x_l = 2$ and $x_r = 1 \Rightarrow x_m = \dfrac{x_r + x_l}{2} \Rightarrow x_m = \dfrac{1+2}{2} = 1.5$.

Check the sign of $f(x_m) \Rightarrow f(1.5) = \ln(1.5) - 2(1.5) + 3 = 0.40547$.

Since $f(x_m) > 0$, then for the next iteration $x_l = 2$ (unchanged) and $x_r = x_m = 1.5$.

<u>2^{nd} iteration:</u> $x_l = 2$ and $x_r = 1.5$.

The calculations must be repeated as in the 1^{st} iteration and continued until $\Delta \le \varepsilon$.

i	x_l	x_r	$x_m = \dfrac{x_r + x_l}{2}$	$f(x_m)$	$\Delta = \left\lvert \dfrac{x_r - x_l}{2} \right\rvert$
1	2	1	1.5	0.405…	0.5
2	2	1.5	1.75	0.059....	0.25
3	2	1.75	1.875	- 0.12….	0.125
4	1.875	1.75	1.8125	- 0.03….	0.0625
5	1.8125	1.75	1.78125	0.14….	0.03125
6	1.8125	1.78125	1.79688	- 0.007….	0.015625
7	1.79688	1.78125	1.78906	0.003…	0.00782
8	1.79688	1.78906	1.79297	- 0.002…	0.00391
9	1.79297	1.78906	1.79101	- 0.0007..	0.00196
10	1.79101	1.78906	1.79003	0.002…	$0.00098 < \varepsilon$

After 10 iterations the first approximate root is $x_{root} \approx 1.79003$.

$$y \approx \ln 1.79003 \approx 0.582232 \quad \Rightarrow \quad \text{the first point of intersection is } (1.79003, 0.582232).$$

<u>H.W:</u>

The second point of intersection is (0.048673,-2.9026).

<u>*Note*:</u>

If we want to estimate the number of iterations required to find the above first root to the given accuracy, then:

$$n \geq \frac{\ln\left|\dfrac{x_r - x_l}{\varepsilon}\right|}{\ln 2} \quad \Rightarrow \quad n \geq \frac{\ln\left|\dfrac{1-2}{1\times 10^{-3}}\right|}{\ln 2} \quad \Rightarrow \quad n \geq 9.96.$$

So we need 10 iterations.

2- Fixed point Method

A fixed point of a function $g(x)$ is a real number p such that $p = g(p)$.

Graphically, fixed points of a function $y = g(x)$ are the points of intersection of $y = g(x)$ and $y = x$.

Fixed point method is used to determine roots of a function $f(x)$ as follows:

1- Rearrange the equation $f(x) = 0$ in the form $x = g(x)$ (so that x is on the left hand side of the equation).

2- Estimate an initial value to the root x_i and substitute it into $g(x)$ to get $g(x_i)$.

3- An improved estimation of the root is determined from $x_{i+1} = g(x_i)$ and so on.

Notes:

1- Fixed point method has very slow convergence.

2- For determining an expected root, lies in the interval (*a*, *b*), a certain expression of $x = g(x)$ seems to converge to this root if the absolute value of the slope of $g(x)$ is less than the slope of $y = x$, that is $|g'(x)| \leq 1$ for all $x \in (a,b)$.

3- A certain expression of $x = g(x)$ may converge to one root at more.

4- If we can not get an expression of the form $x = g(x)$, then we could add x to both sides. For example, we can rewrite the equation $\sin x = 0$ in the form $x = \sin x + x$.

<u>Example 1</u>: Find the maximum value of the function $y = x^3/3 - 1.1x^2 - 3.1x$ correct to three decimals.

Solution:

Maximum value of the function y occurs when $y' = 0$,

$y' = x^2 - 2.2x - 3.1$,

Put $y' = 0 \quad \Rightarrow \quad x^2 - 2.2x - 3.1 = 0 \quad \Rightarrow \quad f(x) = 0$ (Root finding problem)

So we must find the root(s) of $f(x)$ where $f(x) = x^2 - 2.2x - 3.1$.

Check the sign of $f(x)$ at different values of x: (not necessary)

x	- 2	- 1	0	1	2	3	4
$f(x)$	5.3	0.1	- 3.1	- 4.3	- 1.3	- 0.7	4.1

There are two roots: The first root lies between x = - 1 and x = 0 and the second root lies between x = 3 and x = 4.

By using fixed point method, rearrange the equation $f(x)=0$ in the form $x=g(x)$:

$$x^2-2.2x-3.1=0,$$

either $\quad x^2=2.2x+3.1 \quad \Rightarrow \quad x=\sqrt{2.2x+3.1}$ (the first expression)

or $\quad x.(x-2.2)=3.1 \quad \Rightarrow \quad x=\dfrac{3.1}{x-2.2}$ (the second expression)

or $\quad 2.2x=x^2-3.1 \quad \Rightarrow \quad x=\dfrac{x^2-3.1}{2.2}$ (the third expression)

* For the first expression $x=\sqrt{2.2x+3.1}$,

Convergence test: (not necessary)

$$g(x)=\sqrt{2.2x+3.1} \quad \Rightarrow \quad g'(x)=\frac{1.1}{\sqrt{2.2x+3.1}},$$

- For the first root which $\in(-1,0)$,

$$|g'(-1)|=\left|\frac{1.1}{\sqrt{2.2(-1)+3.1}}\right|=1.16>1 \text{ Not Ok,} \qquad |g'(0)|=\left|\frac{1.1}{\sqrt{2.2(0)+3.1}}\right|=0.62\le 1 \text{ Ok.}$$

Thus, this expression will not converge to this root.

- For the second root which $\in(3,4)$,

$$|g'(3)|=\left|\frac{1.1}{\sqrt{2.2(3)+3.1}}\right|=0.35\le 1 \text{ Ok,} \qquad |g'(4)|=\left|\frac{1.1}{\sqrt{2.2(4)+3.1}}\right|=0.32\le 1 \text{ Ok.}$$

Thus, this expression will converge to this root.

1st iteration: Let $x_o = 3 \Rightarrow x_1 = g(x_o) \Rightarrow x_1 = g(3) = \sqrt{2.2(3)+3.1} = 3.114482$.

2nd iteration: $x_1 = 3.114482 \Rightarrow x_2 = g(3.114482) = \sqrt{2.2(3.114482)+3.1} = 3.154657$.

The calculations must be repeated and continued until $\Delta \le \varepsilon$.

i	x_i	$x_{i+1} = g(x_i)$	$\Delta_i = \lvert x_{i+1} - x_i \rvert$
0	3	3.114482	0.11….
1	3.114482	3.154657	0.04….
2	3.154657	3.168635	0.01….
3	3.168635	3.173483	8.2×10^{-3}
4	3.173483	3.175163	1.68×10^{-3}
5	3.175163	3.175745	$5.8\times10^{-4} < \varepsilon$

The root is $x_{root} \approx 3.175745$.

$$y(3.175745) = (3.175745)^3/3 - 1.1(3.175745)^2 - 3.1(3.175745) = -4.924442.$$

Notes:

1- Another arrangement for the above table of calculations may be used as below:

i	x_i	$\Delta_i = \lvert x_i - x_{i-1} \rvert$
0	3	-
1	3.114482	0.11….
2	3.154657	0.04….
3	3.168635	0.01….
4	3.173483	8.2×10^{-3}
5	3.175163	1.68×10^{-3}
6	3.175745	$5.8\times10^{-4} < \varepsilon$

2- If we choose another initial values to the root, this expression will always converge to this root which lies in the interval (3,4), for example:

i	x_i	$\Delta = \lvert x_i - x_{i-1} \rvert$
0	- 1	-
1	0.948683	1.94….
2	2.277521	1.32….
3	2.847902	0.57….
4	3.060292	0.21….
5	3.135704	0.07….
6	3.162048	0.02….
7	3.171199	9.15×10^{-3}
8	3.174372	3.17×10^{-3}
9	3.175471	1.1×10^{-3}
10	3.175852	$3.81\times10^{-4}<\varepsilon$

i	x_i	$\Delta = \lvert x_i - x_{i-1} \rvert$
0	7	-
1	4.301163	2.69….
2	3.544370	0.75….
3	3.301153	0.24….
4	3.219089	0.08….
5	3.190924	0.02….
6	3.181200	9.7×10^{-3}
7	3.177836	3.3×10^{-3}
8	3.176671	1.1×10^{-3}
9	3.176268	$4\times10^{-4}<\varepsilon$

For the second expression $x = \dfrac{3.1}{x-2.2}$,

<u>Convergence test:</u> (not necessary)

$$g(x)=\frac{3.1}{x-2.2} \quad \Rightarrow \quad g'(x)=\frac{-3.1}{(x-2.2)^2},$$

- For the first root which $\in(-1,0)$,

$$|g'(-1)|=\left|\frac{-3.1}{(-1-2.2)^2}\right|=0.3\le 1 \text{ Ok,} \qquad |g'(0)|=\left|\frac{-3.1}{(0-2.2)^2}\right|=0.64\le 1 \text{ Ok.}$$

Thus, this expression will converge to this root.

- For the second root which $\in(3,4)$,

$$|g'(3)|=\left|\frac{-3.1}{(3-2.2)^2}\right|=4.8>1 \text{ Not Ok,} \qquad |g'(4)|=\left|\frac{-3.1}{(4-2.2)^2}\right|=0.96\le 1 \text{ Ok.}$$

Thus, this expression will not converge to this root.

<u>1st iteration:</u> Let $x_o=-1 \Rightarrow x_1=g(x_o) \Rightarrow x_1=g(-1)=\frac{3.1}{(-1)-2.2}=-0.96875$.

<u>2nd iteration:</u> $x_1=-0.96875 \Rightarrow x_2=g(-0.96875)=\frac{3.1}{(-0.96875)-2.2}=-0.978304$.

The calculations must be repeated and continued until $\Delta\le\varepsilon$.

i	x_i	$x_{i+1} = g(x_i)$	$\Delta_i = \lvert x_{i+1} - x_i \rvert$
0	- 1	- 0.96875	0.031….
1	- 0.96875	- 0.978304	9.5×10^{-3}
2	- 0.978304	- 0.975363	2.9×10^{-3}
3	- 0.975363	- 0.976266	$9 \times 10^{-4} < \varepsilon$

The root is $x_{root} \approx -0.976266$.

$$y(-0.976266) = (-0.976266)^3/3 - 1.1(-0.976266)^2 - 3.1(-0.976266) = 1.667862.$$

Thus, the maximum value of y is 1.667862, approximately.

Note:

If we want to know which root would the third expression $x = \dfrac{x^2 - 3.1}{2.2}$ converge to, then we could use the convergence test:

$$g(x) = \frac{x^2 - 3.1}{2.2} \quad \Rightarrow \quad g'(x) = \frac{x}{1.1},$$

- For the first root which $\in (-1,0)$,

$$|g'(-1)| = \left|\frac{-1}{1.1}\right| = 0.91 \le 1 \text{ Ok,} \qquad |g'(0)| = \left|\frac{0}{1.1}\right| = 0 \le 1 \text{ Ok.}$$

Thus, this expression will converge to this root.

- For the second root which $\in (3,4)$,

$$|g'(3)| = \left|\frac{3}{1.1}\right| = 2.7 > 1 \text{ Not Ok,} \qquad |g'(4)| = \left|\frac{4}{1.1}\right| = 3.6 > 1 \text{ Not Ok.}$$

Thus, this expression will not converge to this root.

Example 2: Find the value of x which makes the function $f(x)=(2-x)e^{-x/4}$ equal to 1. ($\varepsilon = 1\times 10^{-3}$)

Solution:

$$f(x)=1 \quad \Rightarrow \quad (2-x)e^{-x/4}=1,$$

$$\therefore (2-x)e^{-x/4}-1=0 \quad \Rightarrow \quad h(x)=0. \text{ (Root finding problem)}$$

So we must find the root(s) of $h(x)$ where $h(x)=(2-x)e^{-x/4}-1$.

Check the sign of $h(x)$ at different values of x: (not necessary)

x	- 2	- 1	0	1	2	3
$h(x)$	1.4	1.3	1	- 0.22	- 1	- 1.47

Thus, there is a root lies between x = 0 and x = 1.

By using fixed point method, rearrange the equation $h(x)=0$ in the form $x=g(x)$:

$$(2-x)e^{-x/4}-1=0 \quad\Rightarrow\quad (2-x)e^{-x/4}=1 \quad\Rightarrow\quad 2-x=\frac{1}{e^{-x/4}},$$

$$2-x=e^{x/4} \quad\Rightarrow\quad x=2-e^{x/4}. \quad \text{(in this expression } g(x)=2-e^{x/4}\text{)}$$

<u>1^{st} iteration:</u> Let $x_o=1 \quad\Rightarrow\quad x_1=g(x_o) \quad\Rightarrow\quad x_1=g(1)=2-e^{(1)/4}=0.715975$.

<u>2^{nd} iteration:</u> $x_1=0.715975 \Rightarrow x_2=g(0.715975)=2-e^{(0.715975)/4}=0.803987$.

The calculations must be repeated and continued until $\Delta \le \varepsilon$.

i	x_i	$x_{i+1} = g(x_i)$	$\Delta_i = \left\|x_{i+1} - x_i\right\|$
0	1	0.715975	0.28….
1	0.715975	0.803987	0.08….
2	0.803987	0.777379	0.02….
3	0.777379	0.785485	8.1×10^{-3}
4	0.785485	0.783021	2.4×10^{-3}
5	0.783021	0.783771	$7.5\times10^{-4}<\varepsilon$

The root is $x_{root} \approx 0.783771$.

Note:

If we start with another possible expression of $x = g(x)$ like:

$$(2-x)e^{-x/4} - 1 = 0 \quad \Rightarrow \quad e^{-x/4} = \frac{1}{2-x} \quad \Rightarrow \quad e^{x/4} = 2-x,$$

$$\frac{x}{4} = \ln|2-x| \quad \Rightarrow \quad x = 4\ln|2-x|. \qquad \text{(In this expression } g(x) = 4\ln|2-x|\text{)}$$

Then, we get the following results:

i	x_i	$x_{i+1} = g(x_i)$	$\Delta_i = \lvert x_{i+1} - x_i \rvert$
0	1	0	1
1	0	2.772589	2.77….
2	2.772589	-1.032034	3.03….
3	-1.032034	4.436934	5.46…. (divergence)

Thus, this expression does not converge to the required root. Therefore we must search for another expression of $x = g(x)$.

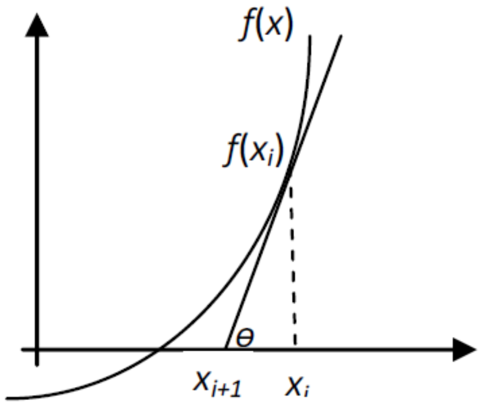
3- Newton-Raphson Method

This is one of the more popular methods used for solving non-linear algebraic equations. It is also known as Newton's method or the tangent method. It is convergent faster than the previous methods. The formula of this method can be derived as follows.

Let x_i be an estimation to the required root of a given function $f(x)$. A better estimation x_{i+1} can be obtained by using the zero of the tangent to the function at x_i. The tangent line passes the x-axis at the improved root x_{i+1}. The value of x_{i+1} can be determined as follows:

$f'(x_i) = \tan\theta$, but from the shown figure:

$$\tan\theta = \frac{f(x_i)}{x_i - x_{i+1}} \quad \Rightarrow \quad \therefore f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}},$$



or $$x_i - x_{i+1} = \frac{f(x_i)}{f'(x_i)} \quad \Rightarrow \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

Notes:

1. Newton-Raphson method has slow convergence in regions of multiple roots.
2. Near the maxima and minima points, Newton-Raphson method is either convergent to these points or convergent to a non-required root or divergent.

Example 1: Find the positive root of $(x^2 - 4\sin x)$ to an accuracy of $\varepsilon = 1\times10^{-6}$.

Solution:

Let $f(x) = x^2 - 4\sin x$, and check the sign of *f(x)*: (not necessary)

x	0	1	2	3
$f(x)$	0	- 2.366	0.363	8.4

There is a positive root lies between $x = 1$ and $x = 2$ and it is closer to $x = 2$.

To find this root by using Newton-Raphson method,

<u>1st iteration:</u> Let $x_o = 2$,

$$f(x) = x^2 - 4\sin x \quad \Rightarrow \quad f(x_o) = f(2) = (2)^2 - 4\sin 2 = 0.362810,$$

$$f'(x) = 2x - 4\cos x \quad \Rightarrow \quad f'(x_o) = f'(2) = 2(2) - 4\cos 2 = 5.664587,$$

$$x_1 = x_o - \frac{f(x_o)}{f'(x_o)} \quad \Rightarrow \quad x_1 = 2 - \frac{0.362810}{5.664587} = 1.935951.$$

<u>2^{nd} iteration:</u> $\quad x_1 = 1.935951.$

The calculations must be repeated as in the 1^{st} iteration and continued until $\Delta \leq \varepsilon$.

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$	$\Delta_i = \lvert x_{i+1} - x_i \rvert$
0	2	0.362810	5.664587	1.935951	0.064….
1	1.935951	0.011623	5.300277	1.933756	2.2×10^{-3}
2	1.933756	1.18×10^{-5}	5.287682	1.933754	2.2×10^{-6}
3	1.933754	1.25×10^{-6}	5.287671	1.933754	$\approx 2.3 \times 10^{-7} < \varepsilon$

After 4 iterations the positive root is $x_{root} \approx 1.933754$.

Note:

Another arrangement for the above table of calculations may be used as below:

i	x_i	$x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)} = x_i - \dfrac{x_i^2 - 4\sin x_i}{2x_i - 4\cos x_i}$	$\Delta_i = \lvert x_{i+1} - x_i \rvert$
0	2	1.935951	0.064….
1	1.935951	1.933756	2.2×10^{-3}
2	1.933756	1.933754	2.2×10^{-6}
3	1.933754	1.933754	$\approx 2.3\times10^{-7} < \varepsilon$

<u>Example 2</u>: Find the root of $f(x)=(2-x)e^{-x/4}-1$ such that $|f(x)|<1\times10^{-6}$.

Solution:

By using Newton-Raphson method,

$$f(x)=(2-x)e^{-x/4}-1,$$

$$f'(x)=(2-x)e^{-x/4}(-1/4)+e^{-x/4}(-1) \quad\Rightarrow\quad f'(x)=(\frac{x}{4}-\frac{3}{2})e^{-x/4}.$$

<u>1st iteration:</u> Let $x_o=3$, (chosen arbitrary)

$$f(x_o)=f(3)=(2-3)e^{-3/4}-1=-1.472366,$$

$$f'(x_o)=f'(3)=(\frac{3}{4}-\frac{3}{2})e^{-3/4}=-0.354275,$$

$$x_1 = x_o - \frac{f(x_o)}{f'(x_o)} \qquad \Rightarrow \qquad x_1 = 3 - \frac{-1.472366}{-0.354275} = -1.156000.$$

<u>2^{nd} iteration:</u> $\quad x_1 = -1.156000.$

The calculations must be repeated as above and continued until $|f(x)| < 1\times10^{-6}$.

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
0	3	- 1.472366	- 0.354275	- 1.156000
1	- 1.156000	3.213550	- 2.388479	0.189438
2	0.189438	0.726814	- 1.385448	0.714043
3	0.714043	0.075722	- 1.105445	0.782542
4	0.782542	0.001130	- 1.072594	0.783596
5	0.783596	$3.4\times10^{-8} < 1\times10^{-6}$		

Hence the root is $x_{root} \approx 0.783596.$

Note:

If we choose $x_o = 8 \;\Rightarrow\; x_1 = 34.778112 \;\Rightarrow\; x_2 = 869.152844.$ (divergence)

3- Numerical Solution of Set of Algebraic Equations

Introduction

The solution of set of algebraic equations is an important step in wide variety of engineering problems, such as the numerical solution of differential equations, the structural analysis, network analysis, ….etc.

Iterative methods

In the these methods an initial set of values of the unknowns are assumed to determine improved approximate values of these unknowns which in turn are used to determine better approximations and so on. This iteration continues until sufficiently values are obtained.

Solution of Set of linear algebraic equations

1- Jacobi iteration

The system of equations:

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1m}x_m = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2m}x_m = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots \quad (1)$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mm}x_m = b_m$$

Can be written as:

$$x_1 = [b_1 - (a_{12}x_2 + \ldots\ldots\ldots + a_{1m}x_m)]/a_{11}$$
$$x_2 = [b_2 - (a_{21}x_1 + \ldots\ldots\ldots + a_{2m}x_m)]/a_{22}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots \quad (2)$$
$$x_m = [b_m - (a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots)]/a_{mm}$$

In this method initial trial values are assumed which are substituted in the iterative equations (Eq.2) of the unknowns to obtain better approximations of the unknowns that are used to obtain new improved approximations. This method converges if :

$$\left|a_{ii}\right| > \sum_{j=1}^{n} \left|a_{ij}\right| \qquad j = 1,2,\ldots\ldots n \text{ but } i \neq j \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

i.e. the absolute value of the element located on the main diagonal in each row is greater than the sum of the absolute values of the other elements in that row. So the procedure of solution in Jacobi method is as follows:

1- The equations are rearranged for condition of convergence in Eq.3.

2- The resulting equations are written in the iterative expressions of Eq.2.

3- A set of initial values of the unknowns are assumed.

4- These values are substituted in the iterative equations to obtain new values.

5- Step 3 is repeated until the required accuracy is achieved.

Example 1: Solve the following set of equations:

$$4x - 8y + z + 21 = 0,$$
$$-2x + y + 5z - 15 = 0,$$
$$4x - y + z - 7 = 0.$$

Solution:.

Use Jacobi iteration,

Step 1: Rearrange the equations for convergence:

$$4x - y + z = 7,$$
$$4x - 8y + z = -21,$$
$$-2x + y + 5z = 15.$$

Step 2: Find the iterative equations:

$$x_{i+1} = (7 + y_i - z_i)/4,$$
$$y_{i+1} = (21 + 4x_i + z_i)/8,$$
$$z_{i+1} = (15 + 2x_i - y_i)/5.$$

Step 3: Assume initial values:

$$x_o = y_o = z_o = 1.$$

Step 4: Substitute the initial values into the iterative equations to get new values:

1st iteration:

$$x_1 = (7 + 1 - 1)/4 = 1.75,$$

$$y_1 = (21 + 4(1) + 1)/8 = 3.25,$$

$$z_1 = (15 + 2(1) - 1)/5 = 3.2.$$

2nd iteration: $x_1 = 1.75$, $y_1 = 3.25$, and $z_1 = 3.2$.

The calculations must be repeated as in the 1st iteration and continued until the required accuracy (if any) is achieved.

No. of Iteration (i)	x_i	y_i	z_i
0	1	1	1
1	1.75	3.25	3.2
2	1.7625	3.9	3.05
3	1.9625	3.8875	2.925
4	1.990625	3.971875	3.0075
5	1.99109	3.99625	3.001875
…..	…..	…..	…..
…..	…..	…..	…..
	$\to 2$	$\to 4$	$\to 3$

2- Gauss-Seidel iteration

As $(x_1)_{i+1}$ is expected to be a better approximation than $(x_1)_i$,then it appears more advantageous to use the value of $(x_1)_{i+1}$ in determining $(x_2)_{i+1}$ rather than using $(x_1)_i$. Similarly, the value of $(x_1)_{i+1}$ and $(x_2)_{i+1}$ are used to determine the value of $(x_3)_{i+1}$, and so on. The using of this procedure will , in general, yield results that are more rapidly convergent than the conventional Jacobi iteration.

Example: Solve the following set of equations:

$$4x - 8y + z + 21 = 0,$$
$$-2x + y + 5z - 15 = 0,$$
$$4x - y + z - 7 = 0.$$

Solution:.

Use Gauss-Seidel iteration,

Step 1: Rearrange the equations for convergence:

$$4x - y + z = 7,$$
$$4x - 8y + z = -21,$$
$$-2x + y + 5z = 15.$$

Step 2: Find the iterative equations:

$$x_{i+1} = (7 + y_i - z_i)/4,$$
$$y_{i+1} = (21 + 4x_{i+1} + z_i)/8,$$
$$z_{i+1} = (15 + 2x_{i+1} - y_{i+1})/5.$$

Step 3: Assume initial values:

$$x_o = y_o = z_o = 1.$$

Step 4: Substitute the initial values into the iterative equations to get new values:

1st iteration:

$$x_1 = (7 + 1 - 1)/4 = 1.75,$$
$$y_1 = (21 + 4(1.75) + 1)/8 = 3.625,$$
$$z_1 = (15 + 2(1.75) - 3.625)/5 = 2.975.$$

<u>2nd iteration:</u> $x_1 = 1.75$, $y_1 = 3.625$, and $z_1 = 2.975$.

The calculations must be repeated as in the 1st iteration and continued until the required accuracy (if any) is achieved.

i	x_i	y_i	z_i
0	1	1	1
1	1.75	3.625	2.975
2	1.9125	3.953125	2.974375
3	1.994688	3.994141	2.999047
…..	…..	…..	…..
…..	…..	…..	…..
	$\rightarrow 2$	$\rightarrow 4$	$\rightarrow 3$

Solution of Set of nonlinear algebraic equations

These equations can be solved by the Gauss- Seidel iteration.

Example 1: Solve the following system:

$$x + 4y + z^2 - 18 = 0\,,$$

$$x^2 + y + 4z - 15 = 0\,,$$

$$4x + y^2 + z - 11 = 0\,.$$

Solution:

Use Gauss-Seidel iteration,

Step 1: Rearrange the equations for convergence:

$$4x + y^2 + z = 11,$$

$$x + 4y + z^2 = 18,$$

$$x^2 + y + 4z = 15.$$

Step 2: Find the iterative equations:

$$x_{i+1} = (11 - y^2_i - z_i)/4,$$

$$y_{i+1} = (18 - x_{i+1} - z^2_i)/4,$$

$$z_{i+1} = (15 - x^2_{i+1} - y_{i+1})/4.$$

Step 3: Assume initial values:

$$x_o = y_o = z_o = 1.$$

Step 4: Substitute the initial values into the iterative equations to get new values:

1st iteration:

$$x_1 = (11 - 1^2 - 1)/4 = 2.25,$$

$$y_1 = (18 - 2.25 - 1^2)/4 = 3.6875,$$

$$z_1 = (15 - 2.25^2 - 3.6875)/4 = 1.5625.$$

2nd iteration: $x_1 = 2.25$, $y_1 = 3.6875$, and $z_1 = 1.562$.

The calculations must be repeated as in the 1st iteration and continued until the required accuracy (if any) is achieved.

No. of Iteration (i)	x_i	y_i	z_i
0	1	1	1
1	2.25	3.6875	1.5625
…..	…..	…..	…..
…..	…..	…..	…..
…..	…..	…..	…..
	$\to 1$	$\to 2$	$\to 3$

Example 2: Solve:

$$x^2 + xy = 10,$$

$$y + 3xy^2 = 57.$$

Solution:

Use the concept of Gauss-Seidel iteration,

Find the iterative equations:

$$x_{i+1} = \sqrt{10 - x_i y_i},$$

$$y_{i+1} = \sqrt{\frac{57 - y_i}{3x_{i+1}}}.$$

Assume initial values:

$$x_o = y_o = 1.$$

Step 4: Substitute the initial values into the iterative equations to get new values:

1st iteration:

$$x_1 = \sqrt{10-(1)(1)} = 3,$$

$$y_1 = \sqrt{\frac{57-1}{3(3)}} = 2.494438.$$

2nd iteration: $x_1 = 3,\ y_1 = 2.494438.$

The calculations must be repeated as in the 1st iteration and continued until the required accuracy (if any) is achieved.

No. of Iteration (i)	x_i	y_i
0	1	1
1	3	2.494438
2	1.586407	3.384172
3	2.152052	2.881771
4	1.948917	3.042387
5	2.017583	2.985723
…..	…..	…..
…..	…..	…..
	$\to 2$	$\to 3$

Note: Another expressions for the iterative equations must be used if divergence is occurred.

6- Numerical Integration

Introduction

The primary purpose of numerical integration (also called quadrature) is the evaluation of integrals which are either impossible or else very difficult to evaluate analytically. Numerical integration is also essential in the evaluation of integrals of functions available only at discrete points. Such functions often result from the numerical solution of differential equations or from experimental data taken at discrete intervals.

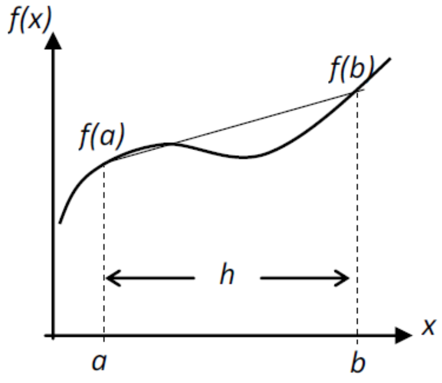
An integral of a given function represents the area enclosed by this function and the x-axis. So, evaluating this area is equivalent to evaluate the integral of this function. In the following, some of numerical techniques, which are used to evaluate an integral, are presented.

f(x)

1- Trapezoidal rule

Consider the integral:

$$I = \int_a^b f(x)dx\,,$$



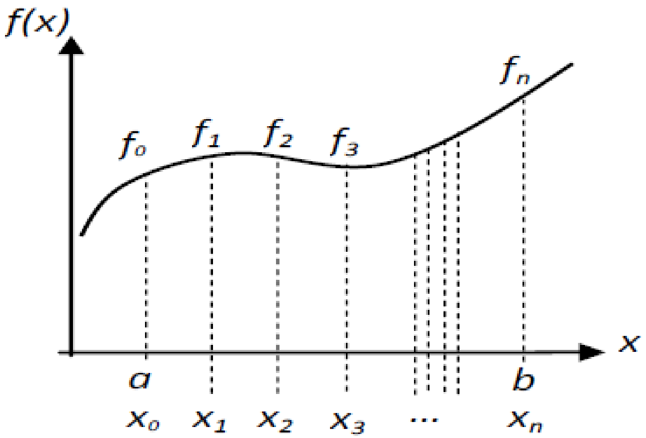
If $f(x)$ is replaced by a straight line (1[st] order polynomial) connecting two points, then the area under this function can be computed from:

$$I = \frac{h}{2}.(f_a + f_b)\,.$$ [trapezoidal rule for one segment (panel)]

If we divide the interval $[a,b]$ into n equal subintervals (segments) then:

$$h = \Delta x = \frac{b-a}{n},$$

$$I = \frac{h}{2}.(f_0 + f_1) + \frac{h}{2}.(f_1 + f_2) + \dots\dots + \frac{h}{2}.(f_{n-1} + f_n),$$



or $$I = \frac{h}{2}.(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n),$$ (trapezoidal rule for n segments)

where, $f_0 = f_a = f(a)$ and $f_n = f_b = f(b)$.

<u>*Notes*</u>:

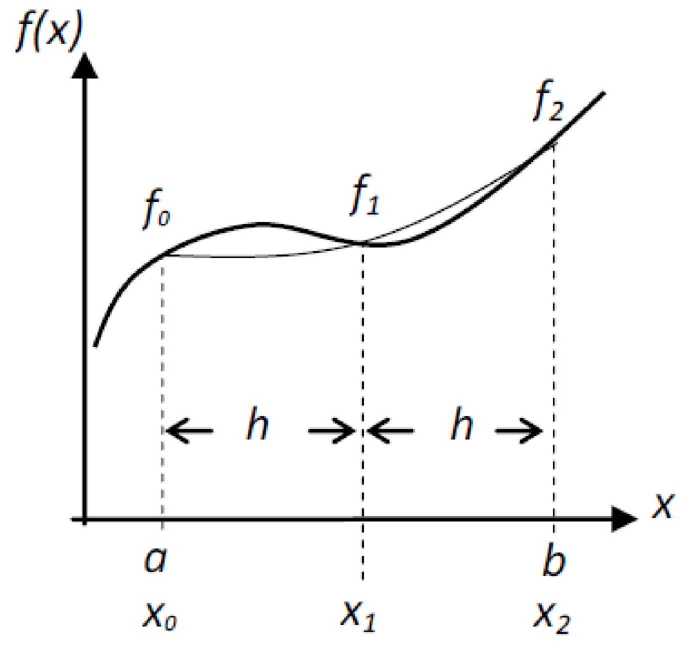
1- The trapezoidal rule gives an answer with an error of order $O(h)^2$.

2- The trapezoidal rule gives an answer which is exact for 1st degree polynomial and approximate for other polynomials of higher degree.

3- Reducing h will, in general, provide more accurate answers.

2- Simpson's rule

2.1- Simpson's 1/3 rule



If $f(x)$ is replaced by a 2nd order polynomial (parabola) connecting three points, then the area under this function can be computed from:

$$I = \frac{h}{3}.(f_0 + 4f_1 + f_2).\text{(Simpson's 1/3 rule for two segments)}$$

If we divide the interval $[a,b]$ into n equal subintervals (n is even) then:

$$I = \frac{h}{3}.(f_0 + 4f_1 + f_2) + \frac{h}{3}.(f_2 + 4f_3 + f_4) + \ldots\ldots\ldots + \frac{h}{3}.(f_{n-2} + 4f_{n-1} + f_n),$$

or $$I = \frac{h}{3}.(f_0 + 4\sum_{i=1,3,5,..}^{n-1} f_i + 2\sum_{i=2,4,6,..}^{n-2} f_i + f_n).$$ [Simpson's 1/3 rule for n (even) segments]

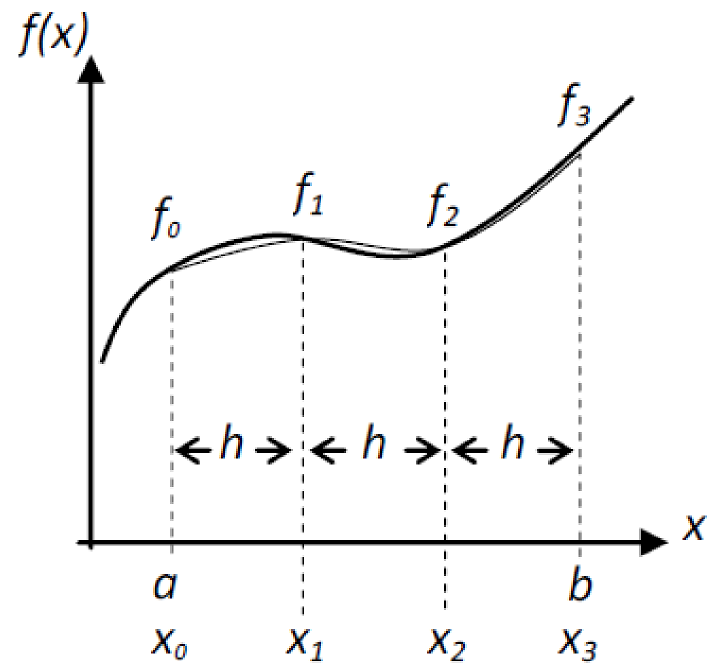
Notes:

1- Simpson's 1/3 rule gives answers with an error of order $O(h)^4$.

2- Simpson's 1/3 gives answers which are exact for polynomials of 2nd degree or lower and approximate for other polynomials of higher degree.

lower and approximate for other polynomials of higher degree.

2.2- Simpson's 3/8 rule



If $f(x)$ is replaced by a 3rd order polynomial (cubic equation) connecting four points, then the area under this function can be computed from:

$$I = \frac{3h}{8}.(f_0 + 3f_1 + 3f_2 + f_3). \qquad \text{(Simpson's 3/8 rule for three segments)}$$

If we divide the interval $[a,b]$ into n equal subintervals (segments) then:

$$I=\frac{3h}{8}.(f_0+3f_1+3f_2+f_3)+\frac{3h}{8}.(f_3+3f_4+3f_5+f_6)+..+\frac{3h}{8}.(f_{n-3}+3f_{n-2}+3f_{n-1}+f_n)$$

or $$I=\frac{3h}{8}.[f_0+3(f_1+f_2+f_4+f_5+\ldots)+2\sum_{i=3,6,9,..}^{n-4}f_i+f_n].$$ (3/8 rule for n segments)

<u>*Notes*</u>:

1- Simpson's 3/8 rule gives answers with an error of order $O(h)^4$.

2- Simpson's 3/8 gives answers which are exact for polynomials of 3rd degree or lower and approximate for other polynomials of higher degree.

Example 1: Evaluate $I = \int_0^{\pi} \sin x dx$ using six segments. Compare with the exact answer.

Solution:

f_0	f_1	f_2	f_3	f_4	f_5	f_6
0	π/6	2π/6	3π/6	4π/6	5π/6	π

Since $\quad n = 6 \quad \Rightarrow \quad h = \Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{6} = \dfrac{\pi}{6}$.

By using the trapezoidal rule (which is of error of $O(h)^2$) $\Rightarrow I = \dfrac{h}{2}.(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n)$,

$$I = \frac{\pi/6}{2}.\{f_0 + 2(f_1 + f_2 + f_3 + f_4 + f_5) + f_6\},$$

$$= \frac{\pi}{12}[\sin 0 + 2\{\sin(\pi/6) + \sin(2\pi/6) + \sin(3\pi/6) + \sin(4\pi/6) + \sin(5\pi/6)\} + \sin(\pi)]$$

$$\approx 1.954097\,.$$

The exact value is $I = [-\cos x]_0^{\pi} = (-\cos\pi) - (-\cos 0) = \{-(-1)\} - \{-(1)\} = 2$.

Percent relative error $P = \left|\dfrac{exact - approx.}{exact}\right| \times 100 = \left|\dfrac{2 - 1.954097}{2}\right| \times 100 = 2.3\%$.

Notes:

* If we use the Simpson's 1/3 rule (which is of error of $O(h)^4$) then,

$$I = \frac{h}{3}.(f_0 + 4\sum_{i=1,3,5,..}^{n-1} f_i + 2\sum_{i=2,4,..}^{n-2} f_i + f_n),$$

$$I = \frac{\pi/6}{3}.\{f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6\},$$

$$= \frac{\pi}{18}[\sin 0 + 4\{\sin(\pi/6) + \sin(3\pi/6) + \sin(5\pi/6)\} + 2\{\sin(2\pi/6) + \sin(4\pi/6)\} + \sin(\pi)]$$

$$\approx 2.000863.$$

Percent relative error $P = \left|\frac{exact - approx.}{exact}\right| \times 100 = \left|\frac{2 - 2.000863}{2}\right| \times 100 = 0.04\%.$

* If we use the Simpson's 3/8 rule (which is of error of $O(h)^4$) then,

$$I = \frac{3h}{8}.[f_0 + 3(f_1 + f_2 + f_4 + f_5 + \ldots) + 2\sum_{i=3,6,9,\ldots}^{n-4} f_i + f_n],$$

$$I = \frac{3(\pi/6)}{8}.\{f_0 + 3(f_1 + f_2 + f_4 + f_5) + 2(f_3) + f_6\},$$

$$= \frac{\pi}{16}[\sin 0 + 3\{\sin(\pi/6) + \sin(2\pi/6) + \sin(4\pi/6) + \sin(5\pi/6)\} + 2\sin(3\pi/6) + \sin(\pi)]$$

$$\approx 2.000005.$$

Percent relative error $P = \left|\frac{exact - approx.}{exact}\right| \times 100 = \left|\frac{2 - 2.000005}{2}\right| \times 100 = 0.0003\%$.

Example 2: Given the function $f(x)=(x+1)^x$, find $\int_1^{1.2} f(x)dx$ correct to three decimals.

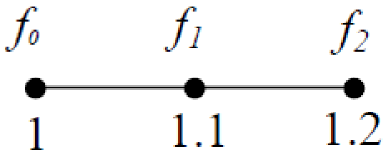
Solution:

By using the trapezoidal rule,

1st iteration: Take $n = 1$ $\Rightarrow$ $h=\dfrac{b-a}{n}=\dfrac{1.2-1}{1}=0.2$,

$$I=\frac{h}{2}.(f_a+f_b)=\frac{h}{2}.[f(1)+f(1.2)]=\frac{0.2}{2}.[(1+1)^1+(1.2+1)^{1.2}]=0.457577.$$

2nd iteration: Take $n = 2$ $\Rightarrow$ $h=\dfrac{b-a}{n}=\dfrac{1.2-1}{2}=0.1$,



$$I=\frac{h}{2}.(f_0+2f_1+f_2)=\frac{h}{2}.[f(1)+2f(1.1)+f(1.2)]=\frac{0.1}{2}.[(1+1)^1+2(1.1+1)^{1.1}+(1.2+1)^{1.2}]=0.454962$$

The calculations must be continued until $\Delta \le \varepsilon$.

No. of Iteration (i)	h_i	I_i	$\Delta_i = \lvert I_i - I_{i-1} \rvert$
1	0.2	0.457577	----
2	0.1	0.454962	2.6×10^{-3}
3	0.05	0.454306	$6.5 \times 10^{-4} < \varepsilon$

$$\therefore \int_{1}^{1.2} f(x)dx \approx 0.454306 .$$

9 9

<u>Example 3</u>: Evaluate $\int_{0}^{9.9} f(x)dx$ using the following data:

x	0	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
$f(x)$	0	0.6	0.8	0.6	0.1	- 0.2	- 0.1	0.1	0.3	0.4

Solution:

Here we have $n = 9$ and $h = 1.1$.

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
0	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9

<u>Solution I</u>: By using the trapezoidal rule $\Rightarrow$ $I = \frac{h}{2}.(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n)$,

$$I = \frac{h}{2}.\{f_0 + 2(f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8) + f_9\},$$

$$= \frac{1.1}{2}[0 + 2\{0.6 + 0.8 + 0.6 + 0.1 + (-0.2) + (-0.1) + 0.1 + 0.3\} + 0.4] = 2.64\,.$$

Solution II: Since $n = 9$ (odd), so we can not use the Simpson's 1/3 rule directly.

Instead, we can apply it for the first 8 segments and the trapezoidal rule for the last segment:

$$I = \frac{h}{3}.\{f_0 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8\} + \frac{h}{2}.(f_8 + f_9),$$

$$= \frac{1.1}{3}[0 + 4\{0.6 + 0.6 + (-0.2) + 0.1\} + 2\{0.8 + 0.1 + (-0.1)\} + 0.3] + \frac{1.1}{2}[0.3 + 0.4] = 2.695 \cdot$$

Solution III: We can apply the Simpson's 1/3 rule for the first 6 segments and the 3/8 rule for the last 3 segments, then:

$$I = \frac{h}{3}.\{f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6\} + \frac{3h}{8}.\{f_6 + 3(f_7 + f_8) + f_9\},$$

$$= \frac{1.1}{3}[0 + 4\{0.6 + 0.6 + (-0.2)\} + 2(0.8 + 0.1) + (-0.1)] + \frac{3(1.1)}{8}[-0.1 + 3(0.1 + 0.3) + 0.4] = 2.70875 \cdot$$

<u>Solution IV</u>: Since $n = 9 = (3 \times 3)$, so we can use the Simpson's 3/8 rule directly:

$$I = \frac{3h}{8}.\{f_0 + 3(f_1 + f_2) + f_3\} + \frac{3h}{8}.\{f_3 + 3(f_4 + f_5) + f_6\} + \frac{3h}{8}.\{f_6 + 3(f_7 + f_8) + f_9\},$$

or $$I = \frac{3h}{8}.\{f_0 + 3(f_1 + f_2 + f_4 + f_5 + f_7 + f_8) + 2(f_3 + f_6) + f_9\},$$

$$= \frac{3(1.1)}{8}[0 + 3\{0.6 + 0.8 + 0.1 + (-0.2) + 0.1 + 0.3\} + 2\{0.6 + (-0.1)\} + 0.4] = 2.68125.$$

4- Taylor Series

Introduction

Taylor series is the foundation of many numerical methods. Many of numerical techniques are derived directly from Taylor series, as are the estimates of the errors involved in employing these techniques.

Maclaurin series

Suppose that the value of the function $f(x)$, shown in the figure, and the values of all of its derivatives at $x=0$, i.e. $f(0), f'(0), f''(0), f'''(0), \ldots$, are known and the value of this function at a point x is to be determined. One method is to approximate $f(x)$ by its tangent line at $x=0$, which has the equation:

$$p(x) = c_0 + c_1 x. \qquad \text{(polynomial of degree 1)}$$

At $x=0$, $p(x) = p(0) \quad \Rightarrow \quad p(0) = c_0 + c_1(0)$,

$\therefore \quad c_0 = p(0)$, but $p(0) = f(0) \Rightarrow \quad c_0 = f(0)$.

$$p'(x) = c_1.$$

At $x=0$, $p'(x) = p'(0) \Rightarrow \quad p'(0) = c_1$,

but $p'(0) = f'(0) \Rightarrow \quad c_1 = f'(0)$.

$\therefore \; p(x) = f(0) + xf'(0) \Rightarrow \; f(x) \approx f(0) + xf'(0)$.

The accuracy of the approximation will be better improved as the degree of the approximation polynomial is increased. If a polynomial of infinite degree is used, then the following approximation is obtained:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \ldots\ldots\ldots,$$

or simply $$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^k(0).$$

The above series (polynomial) is called the Maclaurin series (polynomial).

Taylor series

Maclaurin series gives an approximation of a function $f(x)$ in the vicinity of $x=0$. the more general case of approximating $f(x)$ in the vicinity of an arbitrary value $x=a$ is now considered. The basic idea is the same as before. Thus, if a polynomial of infinite degree is used to approximate a function $f(x)$ which its value and all its derivatives' values are known at $x=a$, then the following polynomial will obtained:

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\frac{(x-a)^4}{4!}f^{IV}(0)+\ldots\ldots,$$

or simply $f(x)=\sum_{k=0}^{\infty}\frac{(x-a)^k}{k!}f^k(a)$.

The above series (polynomial) is called the Taylor series expansion for the function f about $x = a$. It is obvious that Maclaurin series is a special case of Taylor series when the point of expansion is $x = 0$ (i.e. $a = 0$).

Another used formula of Taylor series expansion of a function f about x, where its value and all its derivatives' values are known at the point x, is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{IV}(x) + \ldots\ldots \ .$$

Order of error

The error in the value of $f(x)$ which refers to the error resulted from omitting terms beyond the term contains the n[th] derivative is denoted as $O(x-a)^{n+1}$.

If we take one term of Taylor series, then

$$f(x)=f(a)+O(x-a),$$

if we take two terms, then

$$f(x)=f(a)+(x-a)f'(a)+O(x-a)^2,$$

and if we take n terms, then

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+O(x-a)^n.$$

It is obvious that the error decreases as its order increases, i.e;

$$O(x-a)^{n+1}<O(x-a)^n.$$

Error in truncated Taylor series

The difference $R_n(x)$ (also called the error or remainder) between the exact value of the function $f(x)$ and the value obtained from the n^th^ Taylor series $T_n(x)$ is

$R_n(x) = f(x) - T_n(x)$, which is known as the n^th^ remainder, where

$$T_n(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^k(a) \qquad \text{and} \qquad R_n(x) = \sum_{k=n+1}^{\infty} \frac{(x-a)^k}{k!} f^k(a).$$

Thus the value of $f(x)$ can be written as:

$f(x) = T_n(x) + R_n(x)$, which is called Taylor formula with remainder.

The upper bound of the remainder in a truncated series can be estimated by:

$$R(x) \leq \left| \frac{(x-a)^r}{r!} f^r_{\max} \right| . \qquad \text{(Lagrange's form)}$$

where r is the power of $(x-a)$ in the first truncated term and the maximum value of the derivative f^{r} occurs at some point c lies in the interval $[x,a]$.

Notes:

1- Let the power series is $\sum_{k=0}^{\infty} U_k$. If the limit $f(x) = \lim_{k \to \infty} \left| \frac{U_{k+1}}{U_k} \right| = L$ exists, then

- i- The series converges when $L < 1$.
- ii- The series diverges when $L > 1$.
- iii- The test fails when $L = 1$.

2- The number of terms of a given power series $\sum_{k=0}^{\infty} U_k$, that are required to compute x correct to a given accuracy ε, cab be estimated from $\left| U_k \right| < x.\varepsilon$.

<u>Example 1</u>: Find the Taylor series expansion for the function $f(x)=e^x$ about $x=0$.

Then use it to find $f(x)=e^{0.5}$ to an error of order $O(x)^3$ and compare it with the exact value.

Solution:

When the expansion is about $x=0$, then Taylor series reduces to Maclaurin series.

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots,$$

$$f(x)=e^x \Rightarrow f'(x)=f''(x)=f^n(x)=e^x \Rightarrow f(0)=f'(0)=f''(0)=f^n(0)=e^0=1,$$

$$\therefore \; f(x) = e^x = 1 + x.(1) + \frac{x^2}{2!}.(1) + \frac{x^3}{3!}.(1) + \ldots\ldots\ldots, \quad \text{or } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots.$$

To compute $e^{0.5} \quad \Rightarrow \quad e^x = e^{0.5} \quad \Rightarrow \quad x = 0.5$,

$$\therefore \; e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{2!} + O(0.5)^3 \quad \Rightarrow \quad e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{2} = 1.625.$$

The (exact) value is $e^{0.5} = 1.648721$ (from the scientific calculator).

The percent relative error $P = \left| \dfrac{exact - approx}{exact} \right| \times 100 = \left| \dfrac{1.648721 - 1.625}{1.648721} \right| \times 100 = 1.44\%$.

Example 2: Find the Taylor series expansion of $f(x)=e^{\sqrt{x}}$ about $x=0$.

Solution:

When the expansion is about $x=0$, then Taylor series reduces to Maclaurin series.

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots,$$

$$f(x)=e^{\sqrt{x}} \quad \Rightarrow \quad f(0)=e^{\sqrt{0}}=1,$$

$$f'(x)=e^{\sqrt{x}}.\frac{1}{2}x^{-1/2} \Rightarrow \quad f'(x)=\frac{e^{\sqrt{x}}}{2\sqrt{x}} \quad \Rightarrow \quad f'(0)=\frac{e^{\sqrt{0}}}{2\sqrt{0}}=\frac{1}{0}. \text{ (undefined)}$$

$\because$ Since $f'(0)$ does not exit (undefined) $\Rightarrow$ $\therefore$ $f(x)=e^{\sqrt{x}}$ can not be expanded about $x=0$, or the Taylor series expansion of $f(x)=e^{\sqrt{x}}$ about $x=0$ does not exist.

<u>Example 3</u>: Use Taylor series to determine the square root of 13 to an error of order $O(x)^3$. Estimate the error and compare with the exact value.

Solution:

Taylor series is $f(x) = f(a) + (x-a)f'(a) + \dfrac{(x-a)^2}{2!} f''(a) + O(x-a)^3$.

$\because$ The required is a square root $\Rightarrow$ $\therefore$ Let $f(x) = \sqrt{x}$.

Since the nearest number, of known square root, to 13 is the number 16, so we choose $x = 16$ to determine the square root of any number (i.e. find Taylor series expansion of $f(x) = \sqrt{x}$ about $x = 16$) and then to compute the required square root.

$$f(x) = \sqrt{x} \quad \Rightarrow \quad f(a) = f(16) = \sqrt{16} = 4,$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{2(4)} = \frac{1}{8} \ ,$$

$$f''(x) = \frac{-1}{4\sqrt{x^3}} \quad \Rightarrow \quad f''(16) = \frac{-1}{4\sqrt{16^3}} = \frac{-1}{4\sqrt{16^2.(16)}} = \frac{-1}{4(16)(4)} = \frac{-1}{256} \,,$$

$$\therefore \quad f(x) = \sqrt{x} = 4 + (x-16).\frac{1}{8} + \frac{(x-16)^2}{2}.\frac{-1}{256} + O(x-a)^3 .$$

Now to compute $\sqrt{x} \quad \Rightarrow \quad \sqrt{x} = \sqrt{13} \quad \Rightarrow \quad x = 13,$

$$\therefore \quad \sqrt{13} = 4 + (13-16).\frac{1}{8} + \frac{(13-16)^2}{2}.\frac{-1}{256} = 3.607422 .$$

The error can be estimated from $R(x) \le \left| \frac{(x-a)^r}{r!} f^{r}_{\max} \right| .$

Since the first truncated term in Taylor series contains the 3rd derivative (i.e. $r = 3$),

$$\therefore \; R(x) \le \left| \frac{(13-16)^3}{3!} f'''_{\max} \right|,$$

$$f'''(x) = \frac{-1}{4} \cdot \frac{-3}{2} x^{-5/2} = \frac{3}{8\sqrt{x^5}},$$

$$f'''(x) = f'''(13) = \frac{3}{8\sqrt{13^5}} = 6.15\times10^{-4} \quad \text{and} \quad f'''(a) = f'''(16) = \frac{3}{8\sqrt{16^5}} = 3.66\times10^{-4},$$

$$\therefore f'''_{\max} = 6.15\times10^{-4} \;\Rightarrow\; R(x) \le \left| \frac{(13-16)^3}{6} . 6.15\times10^{-4} \right| \;\Rightarrow\; R(x) \le 2.78\times10^{-3}.$$

The (exact) value is $\sqrt{13} = 3.605551$ (from the hand calculator).

The absolute error $\Delta = \left| exact - approx \right| = \left| 3.60551 - 3.607422 \right| = 1.87\times10^{-3} < 2.78\times10^{-3}$.

Example 4: Find the interval of convergence of Maclaurin expansion for $\sin x$. How many terms are needed to compute $\sin\frac{1}{2}$ accurately to 6 decimals?

Solution:.

Maclaurin series is $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots,$

$$f(x) = \sin x \quad \Rightarrow \quad f(0) = \sin 0 = 0,$$

$$f'(x) = \cos x \quad \Rightarrow \quad f'(0) = \cos 0 = 1,$$

$$f''(x) = -\sin x \quad \Rightarrow \quad f''(0) = -\sin 0 = 0,$$

$$f'''(x) = -\cos x \quad \Rightarrow \quad f'''(0) = -\cos 0 = -1,$$

$$\therefore \; f(x) = \sin x = 0 + x(1) + \frac{x^2}{2!}.(0) + \frac{x^3}{3!}.(-1) + \ldots\ldots \Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots.$$

Convergence test: Check $\lim\limits_{k\to\infty}\left|\dfrac{U_{k+1}}{U_k}\right|$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots \quad \Rightarrow \quad \sin x = \sum_{k=0}^{\infty}(-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!},$$

$$\therefore \lim_{k\to\infty}\left|\frac{U_{k+1}}{U_k}\right| = \lim_{k\to\infty}\left|\frac{\dfrac{x^{2(k+1)+1}}{(2(k+1)+1)!}}{\dfrac{x^{2k+1}}{(2k+1)!}}\right| = \lim_{k\to\infty}\left|\frac{\dfrac{x^{2k+3}}{(2k+3)!}}{\dfrac{x^{2k+1}}{(2k+1)!}}\right|,$$

$$= \lim_{k\to\infty}\left|\frac{x^{2k+3}}{(2k+3)!}\cdot\frac{(2k+1)!}{x^{2k+1}}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+3}}{(2k+3)(2k+2)(2k+1)!}\cdot\frac{(2k+1)!}{x^{2k+1}}\right|,$$

$$= \lim_{k\to\infty}\left|\frac{x^2}{(2k+3)(2k+2)}\right| = \left|\frac{x^2}{(2\infty+3)(2\infty+2)}\right| = \left|\frac{x^2}{\infty}\right| = 0 < 1.$$

$\therefore$ The series is convergent for all values of $x \in R$.

$\therefore$ The interval of convergence is $(-\infty, +\infty)$.

Estimation of terms No.: Use $\left|U_k\right| < x.\varepsilon$,

Here $\varepsilon = 1\times 10^{-6}$ and $\sin x = \sin\frac{1}{2} \Rightarrow x = \frac{1}{2}$,

$$\therefore \quad \frac{\left(\frac{1}{2}\right)^{2k+1}}{(2k+1)!} < \frac{1}{2}.(1\times 10^{-6}) \Rightarrow \frac{1}{2^{2k+1}(2k+1)!} < \frac{1}{2\times 10^{6}},$$

$2^{2k+1}(2k+1)! > 2\times 10^{6} \Rightarrow$ By trial and error $k=4 \Rightarrow \therefore$ we need 5 terms.

7- Numerical Solution of Ordinary Differential Equations

Introduction

An n^{th} order differential equation requires n conditions to obtain a unique solution. If all conditions are specified at the same value of the independent variable, then the problem is called an *initial value problem*, such as

$$y'' + 2y = \ln x, \qquad\qquad y(0) = 1 \text{ and } y'(0) = 0.$$

If the conditions are specified at different values of the independent variable, then it is a *boundary value problem*, such as

$$EIy'' = -M, \qquad\qquad y(0) = 0 \text{ and } y(L) = 0.$$

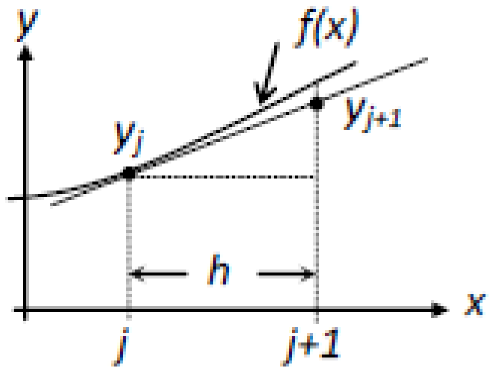
I- Solution of initial value problems

I-a- Solution of 1st order ODEs

Different numerical methods are used to solve 1st ordinary differential equations.

Consider the following 1st order ordinary differential equation $y' = f(x, y)$:

1- Euler's method



From the figure $y'_j = \dfrac{y_{j+1} - y_j}{h}$,

$$\therefore \frac{y_{j+1} - y_j}{h} = f(x_j, y_j),$$

or $\quad y_{j+1} = y_j + h.f(x_j, y_j)$. (New value = old value + step size × slope)

Note: Euler's method gives approximations with an error of 1st order $O(h)$.

2- Second order Runge-Kutta method

$$y_{j+1} = y_j + h.k_2,$$

where $k_1 = f(x_j, y_j)$ and $k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$.

Note: The 2nd order Runge-Kutta method gives approximations with an error of 2nd order $O(h)^2$.

3- Fourth order Runge-Kutta method

$$y_{j+1} = y_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where $\quad k_1 = f(x_j, y_j), \qquad k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1),$

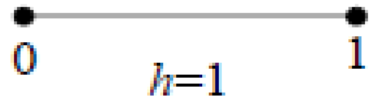
$$k_3 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_2), \text{ and } \quad k_4 = f(x_j + h, y_j + hk_3).$$

Note: The 4^{th} order Runge-Kutta method gives approximations with an error of 4^{th} order $O(h)^4$.

Example 1: Find $y(1)$ if $\dfrac{dy}{dx}=\dfrac{1}{2}(x-y)$, $y(0)=1$. (Use h = 1)

Solution:

The slope $f(x,y)=y'=\dfrac{1}{2}(x-y)$



With the given step size $h=1$, we need one step to move from the start point $x=0$ (where condition is given) to the end point $x=1$ (where y is required).

Solution I: By Euler's method $\Rightarrow$ $y_{j+1}=y_j+h.f(x_j,y_j)$.

$$x_j=0 \quad \text{and} \quad y_j=y(x_j)=y(0)=1,$$

$$\therefore \quad y_1=1+h.f(0,1)=1+(1)[\frac{1}{2}(0-1)]=0.5.$$

* From the analytical solution:

$$y=x-2+3e^{-x/2} \quad \Rightarrow \quad y(1)=1-2+3e^{-1/2}=0.819592 \text{ [the (exact) answer].}$$

* Percent relative error $P=\left|\dfrac{exact-approx.}{exact}\right|\times 100=\left|\dfrac{0.819592-0.5}{0.819592}\right|\times 100\approx 39\%$.

Solution II: By the 2^{nd} order Runge-Kutta method $\Rightarrow$ $y_{j+1} = y_j + h.k_2$,

where $k_1 = f(x_j, y_j)$ and $k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$.

$x_j = 0$ and $y_j = y(x_j) = y(0) = 1$,

$k_1 = f(0,1) = \frac{1}{2}(0-1) = -0.5$,

$$k_2 = f((0+\frac{1}{2}),(1+\frac{1}{2}\times(-0.5))) = f(0.5,0.75) = \frac{1}{2}(0.5-0.75) = -0.125,$$

$$\therefore y_1 = 1 + (1)(-0.125) = 0.875.$$

* Percent relative error $P = \left|\frac{0.819592 - 0.875}{0.819592}\right| \times 100 \approx 6.8\%$.

Solution III: By the 4th order Runge-Kutta method,

$$y_{j+1} = y_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where $k_1 = f(x_j, y_j)$, $\quad k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$,

$k_3 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2}k_2)$, and $\quad k_4 = f(x_j + h, y_j + hk_3)$.

$x_j = 0$ and $\quad y_j = 1$,

$$k_1 = f(0,1) = \frac{1}{2}(0-1) = -0.5,$$

$$k_2 = f((0+\frac{1}{2}),(1+\frac{1}{2}\times(-0.5))) = f(0.5, 0.75) = \frac{1}{2}(0.5-0.75) = -0.125,$$

$$k_3 = f((0+\frac{1}{2}),(1+\frac{1}{2}\times(-0.125))) = f(0.5, 0.9375) = \frac{1}{2}(0.5-0.9375) = -0.21875,$$

$$k_4 = f((0+1),(1+1\times(-0.21875))) = f(1, 0.78125) = \frac{1}{2}(1-0.78125) = 0.109375,$$

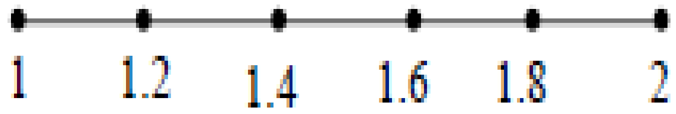
$$\therefore y_1 = 1 + \frac{1}{6}(-0.5 + 2(-0.125) + 2(-0.21875) + 0.109375) = 0.820313.$$

* Percent relative error $P = \left|\frac{0.819592 - 0.820313}{0.819592}\right| \times 100 \approx 0.09\%$.

<u>Example 2</u>: Use Euler's method to find y at $x = 2$, given that

$$dy = e^{x + 0.1y}\,dx, \qquad y(1) = 0. \qquad (\text{Use } h = 0.2)$$

Solution:



$$\frac{dy}{dx} = e^{x + 0.1y} \quad \Rightarrow \quad \text{The slope is } f(x,y) = e^{x + 0.1y}.$$

With the given step size $h = 0.2$, we need 5 steps to move from the start point $x = 1$ (where condition is given) to the end point $x = 2$ (where y is required).

Using Euler's method $\Rightarrow$ $y_{j+1} = y_j + h.f(x_j, y_j)$.

Step 1: $x_j = 1$ and $y_j = y(x_j) = y(1) = 0$,

$$y_{1.2} = y_1 + h.f(1,0) = 0 + 0.2 \times e^{1+0.1(0)} = 0.543656.$$

Step 2: $x_j = 1.2$ and $y_j = 0.543656$,

$$y_{1.4} = y_{1.2} + h.f(1.2, 0.543656) = 0.543656 + 0.2 \times e^{1.2+0.1(0.543656)} = 1.244779.$$

The calculations must be continued for 5 steps.

No. of Step (j)	x_j	y_j	$f(x_j, y_j) = e^{x_j + 0.1 y_j}$	$y_{j+1} = y_j + h.f(x_j, y_j)$
1	1	0	2.718282	0.543656
2	1.2	0.543656	3.505614	1.244779
3	1.4	1.244779	4.592745	2.163328
4	1.6	2.163328	6.149266	3.393181
5	1.8	3.393181	8.493644	5.091910

$\therefore \quad y(2) \approx 5.091910.$

Example 1 For the IVP

$$y' + 2y = 2 - \mathbf{e}^{-4t} \qquad y(0) = 1$$

Use Euler's Method with a step size of $h = 0.1$ to find approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$, and 0.5. Compare them to the exact values of the solution at these points.

Solution

This is a fairly simple linear differential equation so we'll leave it to you to check that the solution is

$$y(t) = 1 + \frac{1}{2}\mathbf{e}^{-4t} - \frac{1}{2}\mathbf{e}^{-2t}$$

In order to use Euler's Method we first need to rewrite the differential equation into the form given in (1).

$$y' = 2 - \mathbf{e}^{-4t} - 2y$$

From this we can see that $f(t, y) = 2 - \mathbf{e}^{-4t} - 2y$. Also note that $t_0 = 0$ and $y_0 = 1$. We can now start doing some computations.

$$f_0 = f(0,1) = 2 - \mathbf{e}^{-4(0)} - 2(1) = -1$$
$$y_1 = y_0 + h\,f_0 = 1 + (0.1)(-1) = 0.9$$

So, the approximation to the solution at $t_1 = 0.1$ is $y_1 = 0.9$.

At the next step we have

$$f_1 = f(0.1, 0.9) = 2 - \mathbf{e}^{-4(0.1)} - 2(0.9) = -0.470320046$$

$$y_2 = y_1 + h\,f_1 = 0.9 + (0.1)(-0.470320046) = 0.852967995$$

Therefore, the approximation to the solution at $t_2 = 0.2$ is $y_2 = 0.852967995$.

I'll leave it to you to check the remainder of these computations.

$$f_2 = -0.155264954 \qquad y_3 = 0.837441500$$

$$f_3 = 0.023922788 \qquad y_4 = 0.839833779$$

$$f_4 = 0.1184359245 \qquad y_5 = 0.851677371$$

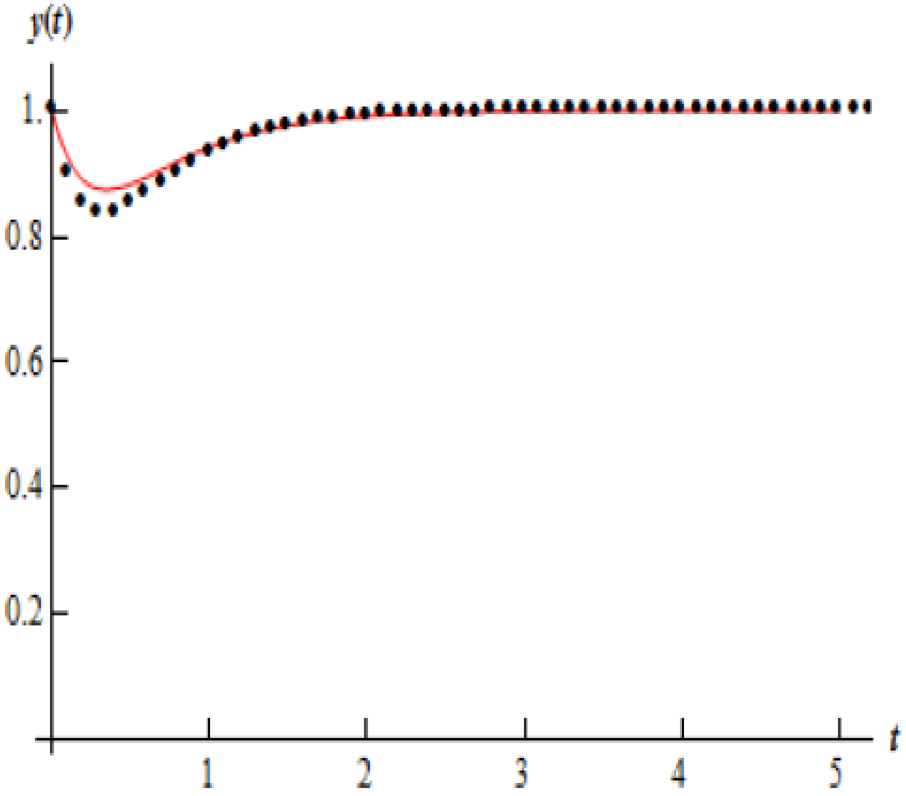
Here's a quick table that gives the approximations as well as the exact value of the solutions at the given points.

Time, t_n	Approximation	Exact	Error
$t_0 = 0$	$y_0 = 1$	$y(0) = 1$	0 %
$t_1 = 0.1$	$y_1 = 0.9$	$y(0.1) = 0.925794646$	2.79 %
$t_2 = 0.2$	$y_2 = 0.852967995$	$y(0.2) = 0.889504459$	4.11 %
$t_3 = 0.3$	$y_3 = 0.837441500$	$y(0.3) = 0.876191288$	4.42 %
$t_4 = 0.4$	$y_4 = 0.839833779$	$y(0.4) = 0.876283777$	4.16 %
$t_5 = 0.5$	$y_5 = 0.851677371$	$y(0.5) = 0.883727921$	3.63 %

We've also included the error as a percentage. It's often easier to see how well an approximation does if you look at percentages. The formula for this is,

$$percent\ error = \frac{\left|exact - approximate\right|}{exact} \times 100$$

We used absolute value in the numerator because we really don't care at this point if the approximation is larger or smaller than the exact. We're only interested in how close the two are.



Example 2 Repeat the previous example only this time give the approximations at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. Use $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$, and $h = 0.001$ for the approximations.

Solution

Below are two tables, one gives approximations to the solution and the other gives the errors for each approximation. We'll leave the computational details to you to check.

Approximations

Time	Exact	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	0.9414902	0.9313244	0.9364698	0.9404994	0.9409957	0.9413914
$t = 2$	0.9910099	0.9913681	0.9911126	0.9910193	0.9910139	0.9910106
$t = 3$	0.9987637	0.9990501	0.9988982	0.9987890	0.9987763	0.9987662
$t = 4$	0.9998323	0.9998976	0.9998657	0.9998390	0.9998357	0.9998330
$t = 5$	0.9999773	0.9999890	0.9999837	0.9999786	0.9999780	0.9999774

Percentage Errors

Time	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	1.08 %	0.53 %	0.105 %	0.053 %	0.0105 %
$t = 2$	0.036 %	0.010 %	0.00094 %	0.00041 %	0.0000703 %
$t = 3$	0.029 %	0.013 %	0.0025 %	0.0013 %	0.00025 %
$t = 4$	0.0065 %	0.0033 %	0.00067 %	0.00034 %	0.000067 %
$t = 5$	0.0012 %	0.00064 %	0.00013 %	0.000068 %	0.000014 %

We can see from these tables that decreasing h does in fact improve the accuracy of the approximation as we expected.

Example 3 For the IVP

$$y' - y = -\frac{1}{2}\mathbf{e}^{\frac{t}{2}}\sin(5t) + 5\mathbf{e}^{\frac{t}{2}}\cos(5t) \qquad y(0) = 0$$

Use Euler's Method to find the approximation to the solution at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. Use $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$, and $h = 0.001$ for the approximations.

Solution

We'll leave it to you to check the details of the solution process. The solution to this linear first order differential equation is.

$$y(t) = \mathbf{e}^{\frac{t}{2}}\sin(5t)$$

Here are two tables giving the approximations and the percentage error for each approximation.

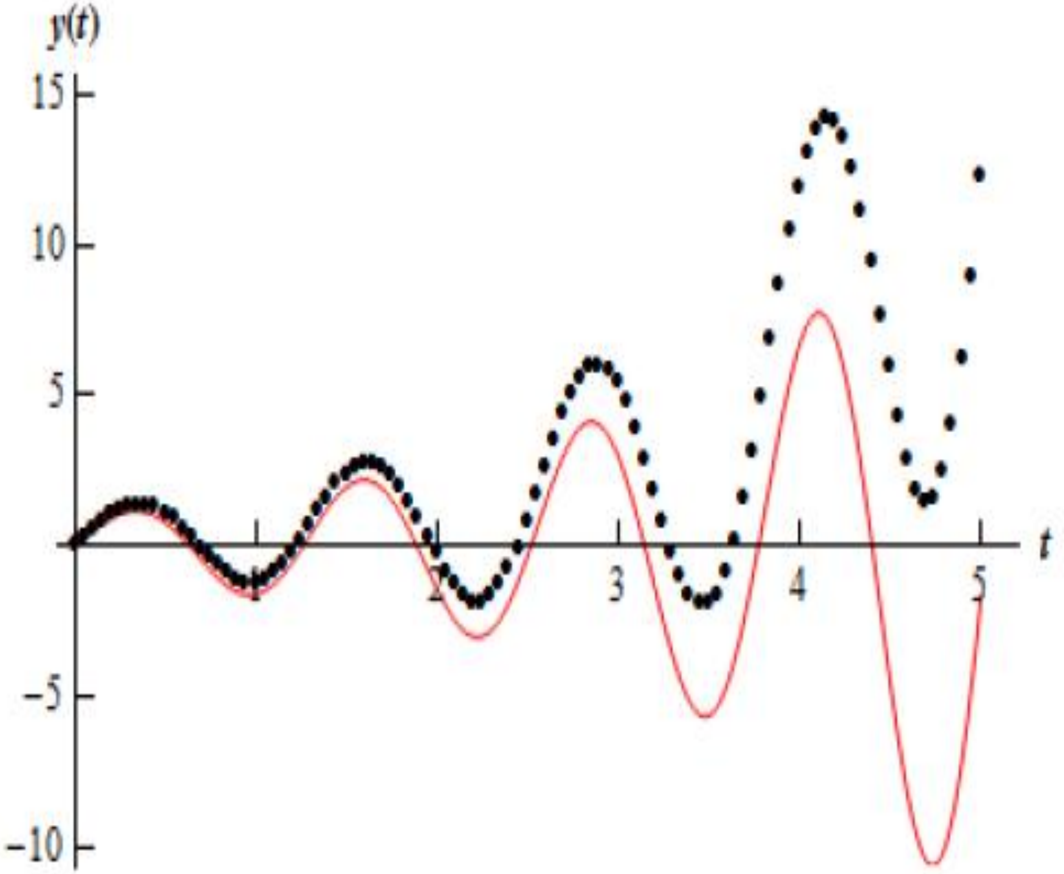
Approximations

Time	Exact	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	-1.58100	-0.97167	-1.26512	-1.51580	-1.54826	-1.57443
$t = 2$	-1.47880	0.65270	-0.34327	-1.23907	-1.35810	-1.45453
$t = 3$	2.91439	7.30209	5.34682	3.44488	3.18259	2.96851
$t = 4$	6.74580	15.56128	11.84839	7.89808	7.33093	6.86429
$t = 5$	-1.61237	21.95465	12.24018	1.56056	0.0018864	-1.28498

Percentage Errors

Time	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	38.54 %	19.98 %	4.12 %	2.07 %	0.42 %
$t = 2$	144.14 %	76.79 %	16.21 %	8.16 %	1.64 %
$t = 3$	150.55 %	83.46 %	18.20 %	9.20 %	1.86 %
$t = 4$	130.68 %	75.64 %	17.08 %	8.67 %	1.76 %
$t = 5$	1461.63 %	859.14 %	196.79 %	100.12 %	20.30 %

Below is a graph of the solution (the line) as well as the approximations (the dots) for $h = 0.05$.



Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds.

Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + f(t_i,\theta_i)h$$

$$\begin{aligned}
\theta_1 &= \theta_0 + f(t_0,\theta_0)h \\
&= 1200 + f(0,1200)240 \\
&= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)240 \\
&= 1200 + (-4.5579)240 \\
&= 106.09K
\end{aligned}$$

θ_1 is the approximate temperature at $t = t_1 = t_0 + h = 0 + 240 = 240$

$$\theta(240) \approx \theta_1 = 106.09K$$

Solution Cont

Step 2: For $i = 1, \quad t_1 = 240, \quad \theta_1 = 106.09$

$$
\begin{aligned}
\theta_2 &= \theta_1 + f(t_1, \theta_1)h \\
&= 106.09 + f(240, 106.09)240 \\
&= 106.09 + \left(-2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right)\right)240 \\
&= 106.09 + (0.017595)240 \\
&= 110.32K
\end{aligned}
$$

θ_2 is the approximate temperature at $t = t_2 = t_1 + h = 240 + 240 = 480$

$$\theta(480) \approx \theta_2 = 110.32K$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.00333\theta\right)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

Comparison of Exact and Numerical Solutions

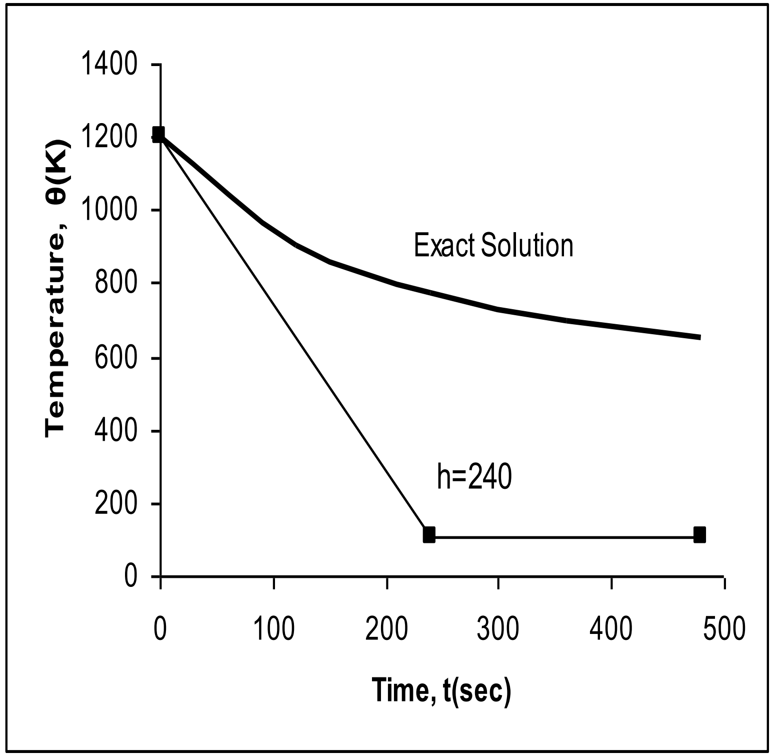


Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step, *h*	θ(480)	E_t	$\|\epsilon_t\|$%
480	−987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.7	14.80	2.286

$\theta(480) = 647.57K$ (exact)

Comparison with exact results

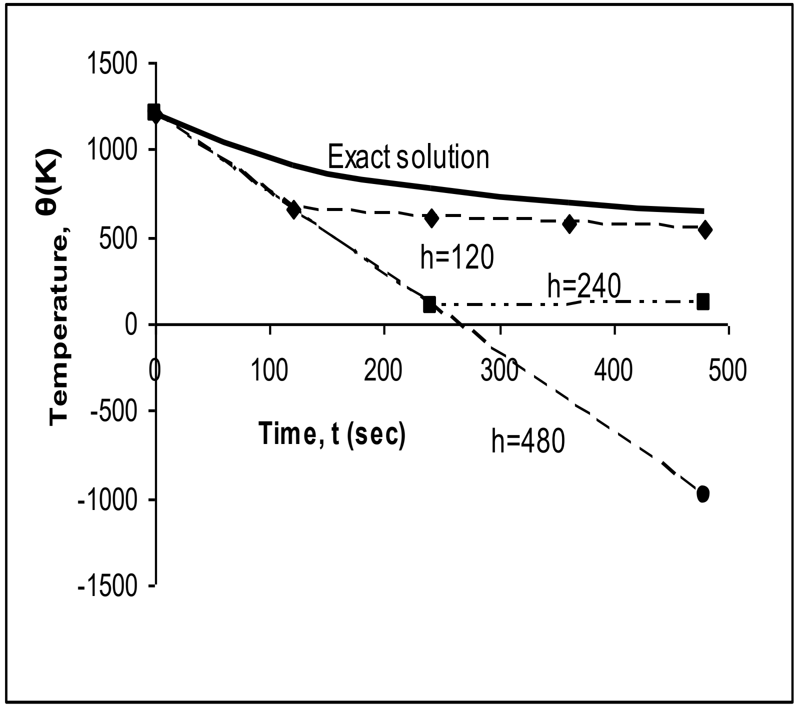


Figure 4. Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler’s Method

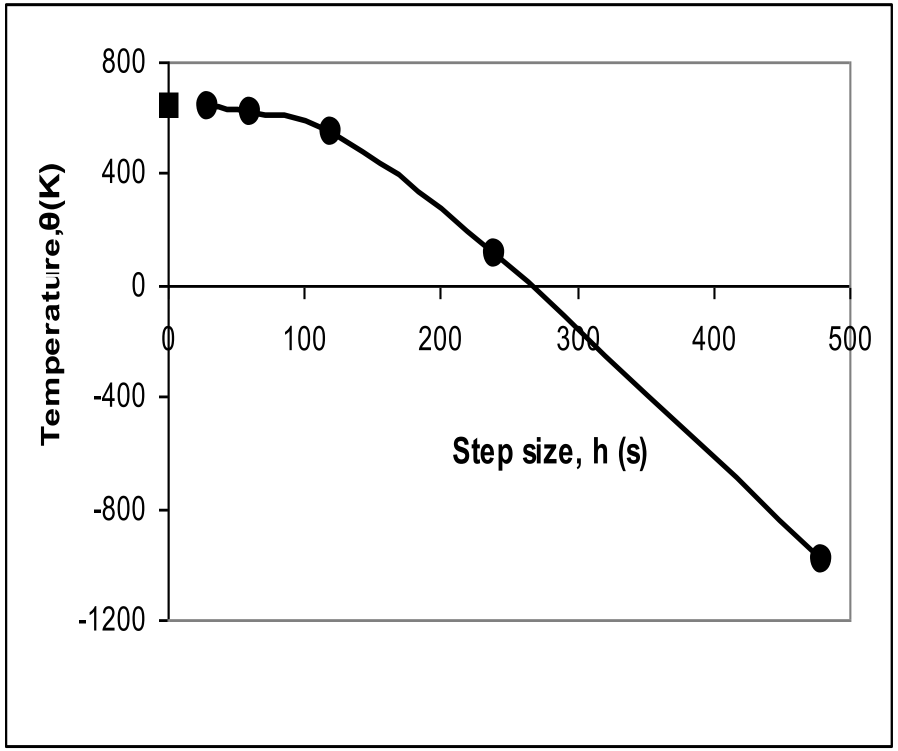


Figure 5. Effect of step size in Euler’s method.

Direct Method of Interpolation

Osama Sahib Jafar

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.

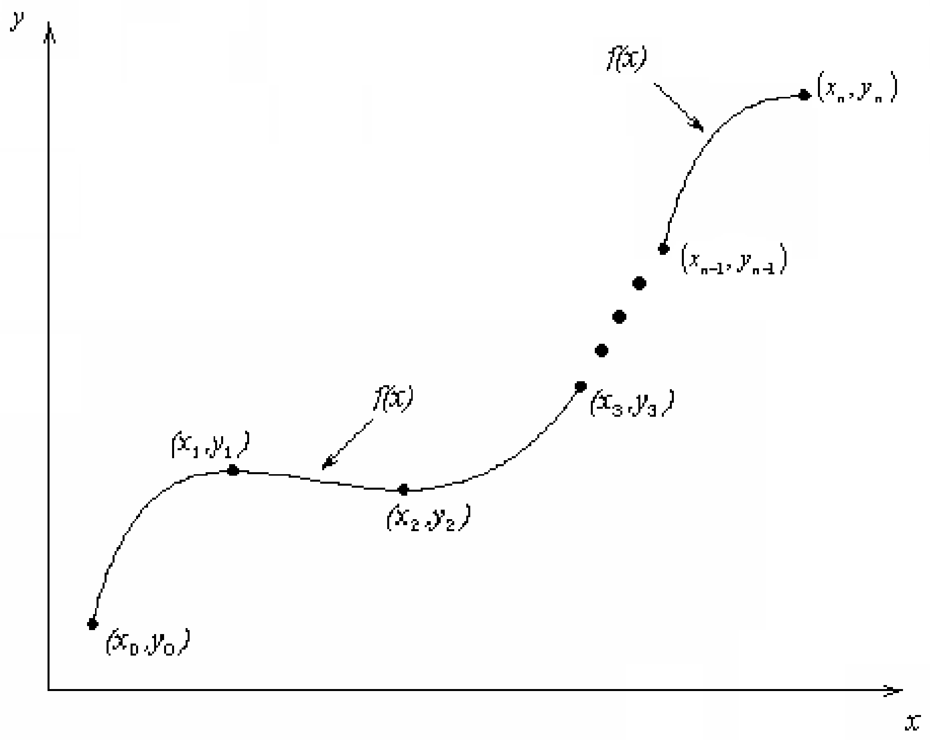


Figure 1 Interpolation of discrete.

Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate

Direct Method

Given 'n+1' data points (x_0,y_0), (x_1,y_1),………….. (x_n,y_n),
pass a polynomial of order 'n' through the data as given
below:

$$y = a_0 + a_1 x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n x^n .$$

where a_0, a_1,………………. a_n are real constants.

- Set up 'n+1' equations to find 'n+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.

Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Find the velocity at t=16 seconds using the direct method for linear interpolation.

Table 1 Velocity as a function of time.

$t, (\text{s})$	$v(t), (\text{m/s})$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

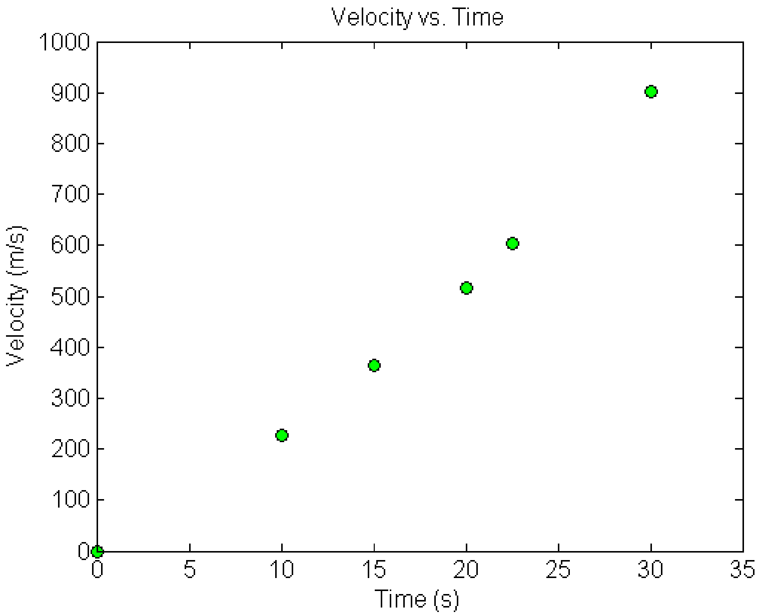


Figure 2 Velocity vs. time data for the rocket example

Linear Interpolation

$$v(t) = a_0 + a_1 t$$

$$v(15) = a_0 + a_1(15) = 362.78$$

$$v(20) = a_0 + a_1(20) = 517.35$$

Solving the above two equations gives,

$$a_0 = -100.93 \qquad a_1 = 30.914$$

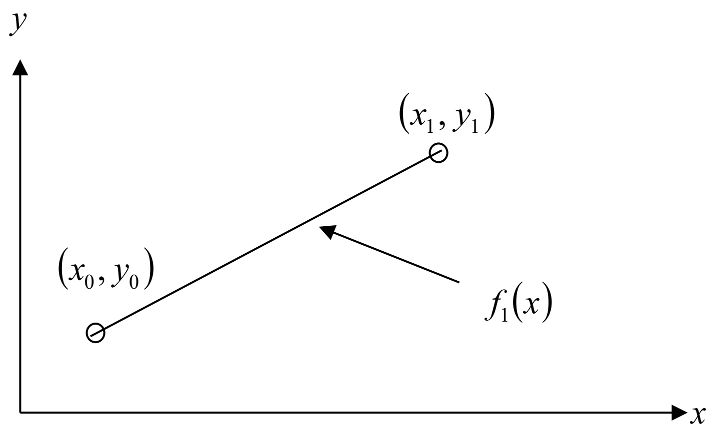


Figure 3 Linear interpolation.

Hence

$$v(t) = -100.93 + 30.914t, \;\; 15 \le t \le 20.$$

$$v(16) = -100.93 + 30.914(16) = 393.7 \text{ m/s}$$

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Find the velocity at t=16 seconds using the direct method for quadratic interpolation.

Table 2 Velocity as a function of time.

$t, (\text{s})$	$v(t), (\text{m/s})$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

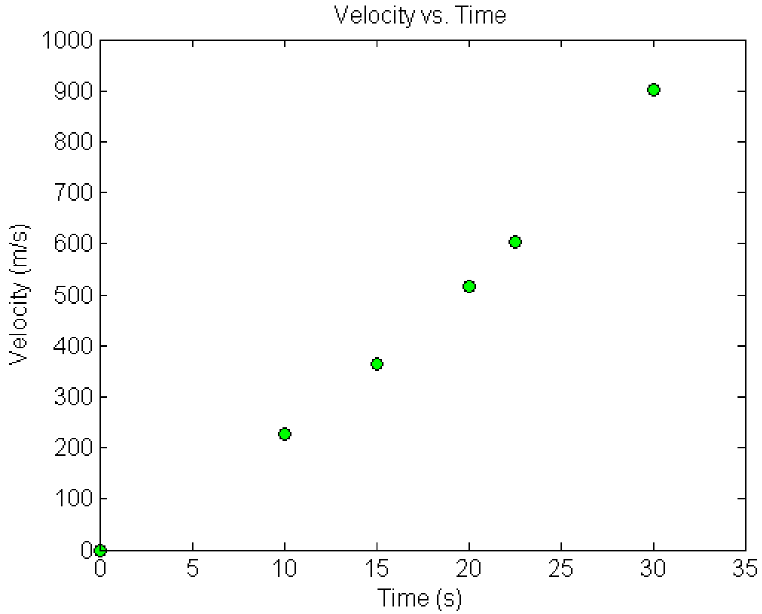


Figure 5 Velocity vs. time data for the rocket example

Quadratic Interpolation

Quadratic Interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2$$

$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$

$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$

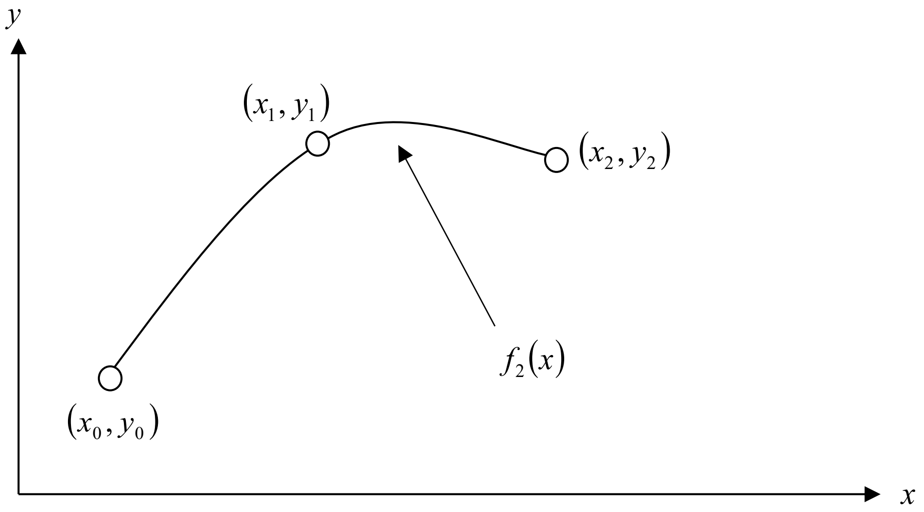


Figure 6 Quadratic interpolation.

Solving the above three equations gives

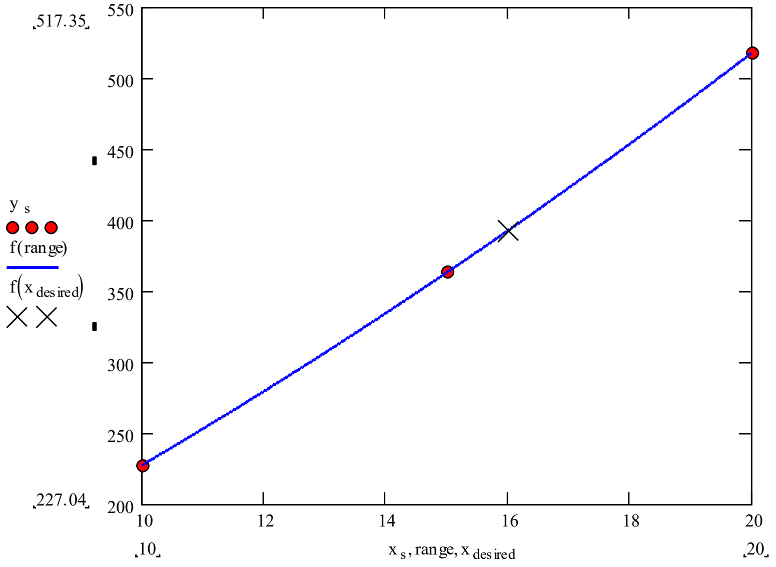
$$a_0 = 12.05 \quad a_1 = 17.733 \quad a_2 = 0.3766$$

Quadratic Interpolation (cont.)

$$v(t) = 12.05 + 17.733t + 0.3766t^2, \ 10 \le t \le 20$$

$$v(16) = 12.05 + 17.733(16) + 0.3766(16)^2$$

$$= 392.19 \text{ m/s}$$



The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$

$$= 0.38410\%$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Find the velocity at t=16 seconds using the direct method for cubic interpolation.

Table 3 Velocity as a function of time.

$t,(\mathrm{s})$	$v(t),(\mathrm{m/s})$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

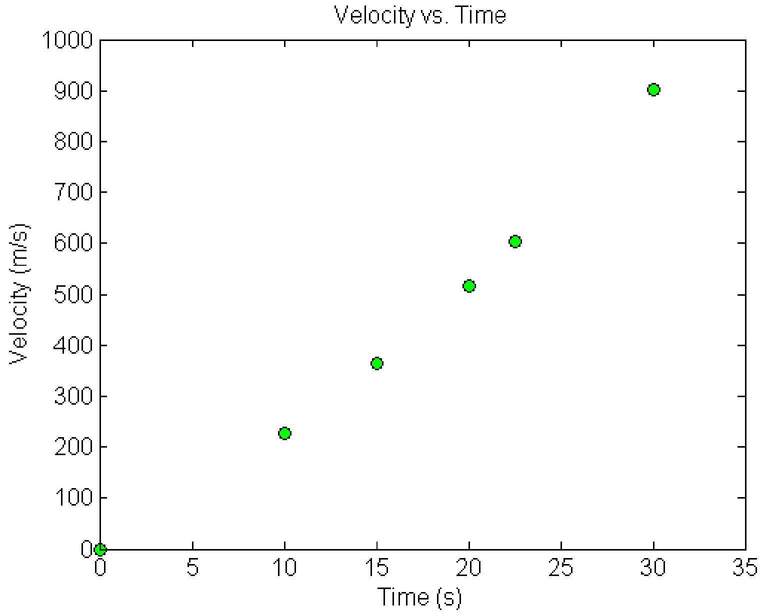


Figure 6 Velocity vs. time data for the rocket example

Cubic Interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$v(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

$$v(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3$$

$$v(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3$$

$$v(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3$$

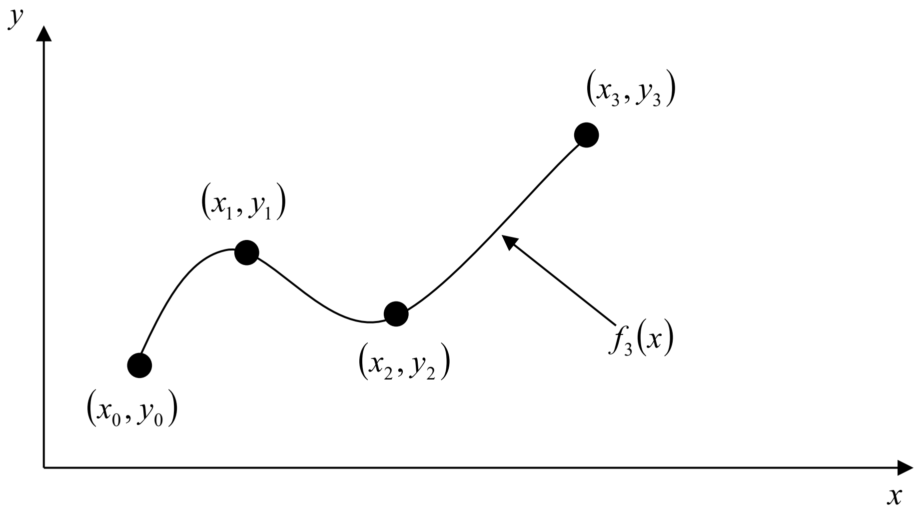


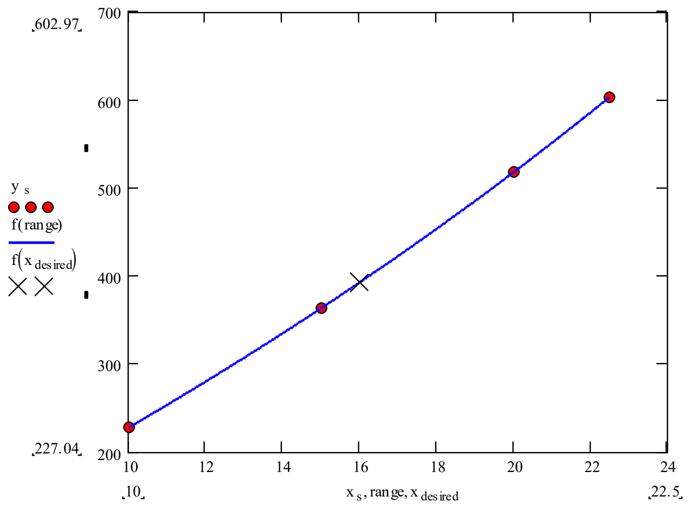
Figure 7 Cubic interpolation.

$$a_0 = -4.2540 \qquad a_1 = 21.266 \qquad a_2 = 0.13204 \qquad a_3 = 0.0054347$$

Cubic Interpolation (contd)

$$v(t) = -4.2540 + 21.266t + 0.13204\,t^2 + 0.0054347\,t^3, \quad 10 \le t \le 22.5$$

$$v(16) = -4.2540 + 21.266(16) + 0.13204(16)^2 + 0.0054347(16)^3$$
$$= 392.06 \text{ m/s}$$



The absolute percentage relative approximate error $|\in_a|$ between second and third order polynomial is

$$|\in_a| = \left|\frac{392.06 - 392.19}{392.06}\right| \times 100$$
$$= 0.033269\%$$

Comparison Table

t(s)	v (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Table 4 Comparison of different orders of the polynomial.

Order of Polynomial	1	2	3
$v(t=16)$ m/s	393.7	392.19	392.06
Absolute Relative Approximate Error	----------	0.38410 %	0.033269 %

Distance from Velocity Profile

Find the distance covered by the rocket from t=11s to t=16s ?

$$v(t) = -4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t)\,dt$$

$$= \int_{11}^{16} \left(-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3\right)dt$$

$$= \left[-4.2540t + 21.266\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16}$$

$$= 1605\,\text{m}$$

Acceleration from Velocity Profile

Find the acceleration of the rocket at t=16s given that

$$v(t) = -4.2540 + 21.266t + 0.13204^{2} + 0.0054347\,t^{3}, 10 \le t \le 22.5$$

$$a(t) = \frac{d}{dt}v(t)$$

$$= \frac{d}{dt}\left(-4.2540 + 21.266t + 0.13204t^{2} + 0.0054347t^{3}\right)$$

$$= 21.289 + 0.26130t + 0.016382t^{2}, \quad 10 \le t \le 22.5$$

$$a(16) = 21.266 + 0.26408(16) + 0.016304(16)^{2}$$

$$= 29.665\ \text{m/s}^{2}$$