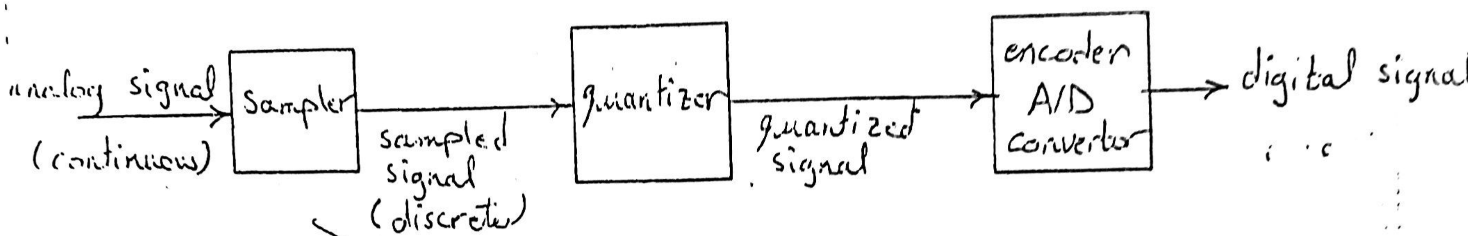
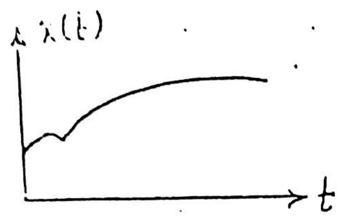
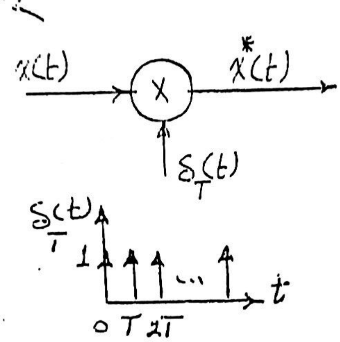
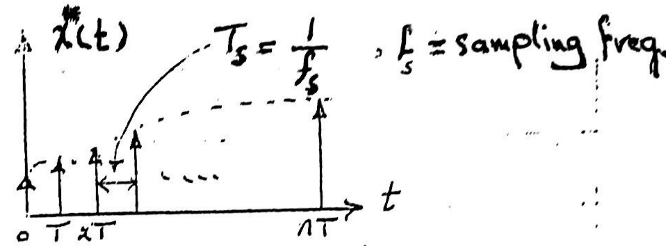
The Z- transform





Continuous sig.





discrete (sampled) sig.

$t \to nT \to n$

The sampled signal is obtained by sampling a continuous (analog) signal at discrete values of time, every T seconds.

The sampler is considered as an impulse amplitude modulator, and the o/p is a train of impulses separated by intervals of T_s (sampling period), and of amplitude of $x(nT)$

$$x^*(t) = \sum_{n=0}^{\infty} x(nT)\ \delta(t-nT)$$

$$= x(0)\ \delta(t) + x(T)\ \delta(t-T) + x(2T)\ \delta(t-2T) + \cdots$$

Taking L.T

$$\mathcal{L}\{x^*(t)\} = X^*(s) = x(0) + x(T)\ e^{-Ts} + x(2T)\ e^{-2Ts} + \cdots = \sum_{n=0}^{\infty} x(nT)\ e^{-nTs}$$

$X^*(s)$ {contains e^{-Ts}} is non algebraic fun. when taking the inverse of L.T. To restor the power of L.T,

$z = e^{j\omega t}$

Z.T is defined

$z = e^{sT}$

$$X^*(s)\Big|_{z=e^{sT}} = \sum_{n=0}^{\infty} x(nT)\, z^{-n}$$

$$\therefore X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

one sided Z.T

two sided $\rightarrow \sum_{-\infty}^{\infty}$

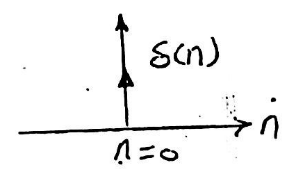
$$x(n) = \mathcal{Z}^{-1}\{X(z)\}$$

inverse of Z.T

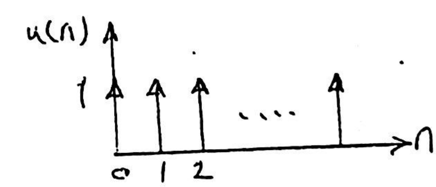
The Z.T is important in dealing with

1. Sampled data systems
2. Discrete " "
3. Digital control "

Z.T of some fun. $t \to nT \to n$



1. $\mathcal{Z}\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n)\, z^{-n} = \underline{1}$



2. $\mathcal{Z}\{u(n)\} = \sum_{n=0}^{\infty} 1\, z^{-n}$

$= 1 + z^{-1} + z^{-2} + \dots$

$= \dfrac{1}{1 - z^{-1}} \qquad |z^{-1}| < 1$

$= \dfrac{z}{z-1}$

by using $\dfrac{1}{1-x} = 1 + x + x^2 + \dots \;, \quad |x| < 1$

ROC

3. $\mathcal{Z}\{e^{-an}\} = \sum_{n=0}^{\infty} e^{-an} z^{-n} = 1 + e^{-a} z^{-1} + e^{-2a} z^{-2} + \dots$

$$= \frac{1}{1 - e^{-a} z^{-1}} = \boxed{\frac{z}{z - e^{-a}}} \quad , \quad |e^{-a} z^{-1}| < 1$$

4. $\mathcal{Z}\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n} = 1 + a z^{-1} + a^2 z^{-2} + \dots$

$$= \frac{1}{1 - a z^{-1}} = \boxed{\frac{z}{z - a}} \quad |a z^{-1}| < 1$$

5. $\mathcal{Z}\{n\} = \sum_{n=0}^{\infty} n z^{-n} = z^{-1} + 2z^{-2} + 3z^{-3}$

$$= z^{-1}\{1 + 2z^{-1} + 3z^{-2} + \dots\}$$

$$= \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z-1)^2}$$

using binomial series; $1 + 2y + 3y^2 + \dots = \frac{1}{(1-y)^2}$ $\quad |y| < 1$

Properties of Z.T :- خواص تحويلات الـ Z

1. Linearity

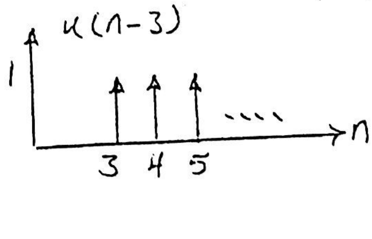
ثوابت

$$\mathcal{Z}\{a_1 x_1(n) \pm a_2 x_2(n)\} = a_1 X_1(z) \pm a_2 X_2(z)$$

Ex $\mathcal{Z}\left\{\frac{1}{a}(1 - e^{-an})\right\} = \frac{1}{a}\left\{\frac{z}{z-1} - \frac{z}{z - e^{-a}}\right\}$

2. Shifting ; " without I.C " → initial condition

$$\mathcal{Z}\{x(n-m)\} = z^{-m} X(z) \quad \to \text{Integer}$$



$$\mathcal{Z}\{x(n-1)\} = z^{-1} X(z)$$

Proof:

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=0}^{\infty} x(n-1)\, z^{-n} = z^{-1} \sum_{n=0}^{\infty} x(n-1)\, z^{-(n-1)}$$

$k = n-1$

$$= z^{-1} \sum_{k=-1}^{\infty} x(k)\, z^{-k} = z^{-1} \sum_{k=0}^{\infty} x(k)\, z^{-k} = z^{-1} X(z)$$

[illegible] قيمة n=0 وباقي الدالة في صفر.

Ex $\mathcal{Z}\{u(n-4)\} = z^{-4}\, \mathcal{Z}\{u(n)\} = z^{-4} \cdot \dfrac{z}{z-1} = \dfrac{z^{-3}}{z-1}$

$\mathcal{Z}\{\delta(n-5)\} = z^{-5} \cdot 1$

3. Multiplication by exponential

$$\mathcal{Z}\{x(n)\, e^{\pm an}\} = X(z\, e^{\mp a})$$

Proof:

$$\mathcal{Z}\{x(n)\, e^{-an}\} = \sum_{n=0}^{\infty} x(n)\, e^{-an}\, z^{-n} = \sum_{n=0}^{\infty} x(n)\, (z e^{a})^{-n} = X(z e^{a})$$

Ex $\mathcal{Z}\{n e^{an}\}$

$\mathcal{Z}\{n\} = \dfrac{z}{(z-1)^2}$

$$\therefore \mathcal{Z}\{n e^{an}\} = \frac{z e^{-a}}{(z e^{-a}-1)^2} = \frac{z e^{a}}{(z-e^{a})^2}$$

4. Multiplication by n

$$\mathcal{Z}\{n\, x(n)\} = -z \frac{d}{dz} X(z).$$

Ex $\mathcal{Z}\{n e^{an}\} = -z \dfrac{d}{dz} \dfrac{z}{z-e^{a}} = \dfrac{z e^{a}}{(z-e^{a})^2}$

5. Initial Value Theorem

$$\lim_{n\to 0} x(n) = \lim_{z\to\infty} X(z) = \lim_{t\to 0} x(t) = x(0)$$

Proof:

$$\mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n} = x(0) + x(1)\underbrace{z^{-1}}_{\frac{1}{z}} + x(2)\underbrace{z^{-2}}_{\frac{1}{z^2}} + \cdots$$

$$\lim_{z\to\infty} X(z) = x(0)$$

6. Final Value Theorem:-

$$\lim_{n\to\infty} x(n) = \lim_{t\to\infty} x(t) = \boxed{x(\infty)} = \boxed{\lim_{z\to 1} (1-z^{-1})\, X(z)}$$

Ex Find the final value of:

$$X(z) = \frac{1}{a}\left(\frac{z}{z-1} - \frac{ze^{a}}{ze^{a}-1}\right)$$

$$\lim_{z\to 1} (1-z^{-1})X(z) = \lim_{z\to 1} \left(\frac{z-1}{z}\right)\cdot \frac{1}{a}\left(\frac{z}{z-1} - \frac{ze^{a}}{ze^{a}-1}\right) = \frac{1}{a}$$

Note: $\lim_{n\to\infty} \frac{1}{a}(1-e^{-an}) = \frac{1}{a}$

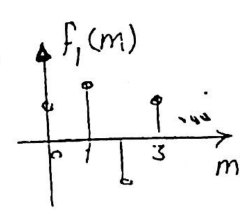
7. Discrete Convolution:-

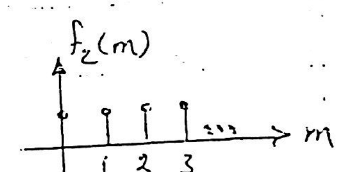
تسمية المطاوي مباشرة . f_k

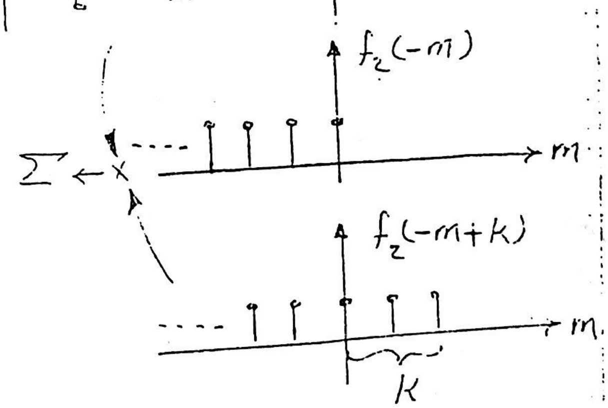
$$f_1(n) \circledast f_2(n) = \sum_{m=0}^{k} f_1(m) f_2(k-m) = \sum_{m=0} f_1(k-m) f_2(m)$$

$$\mathcal{Z}\left\{\sum_{m=0}^{k} f_1(m)\, f_2(k-m)\right\}$$

$$= \mathcal{Z}\left\{\sum_{m=0}^{k} f_1(k-m)\, f_2(m)\right\} = F_1(z)\cdot F_2(z)$$







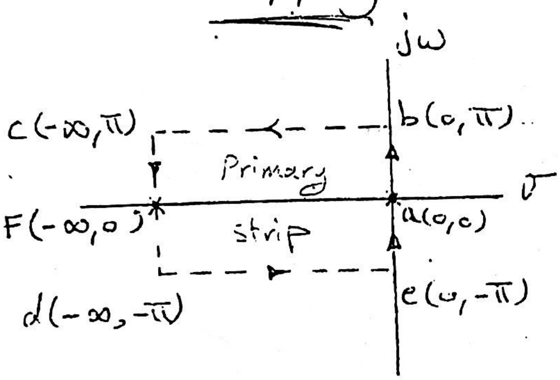
Proof:

$$\mathcal{Z}\left\{\sum_{m=0}^{k} f_1(m) f_2(k-m)\right\} = \sum_{K=0}^{\infty}\left\{\sum_{m=0}^{k} f_1(m) f_2(K-m)\right\} z^{-K}$$

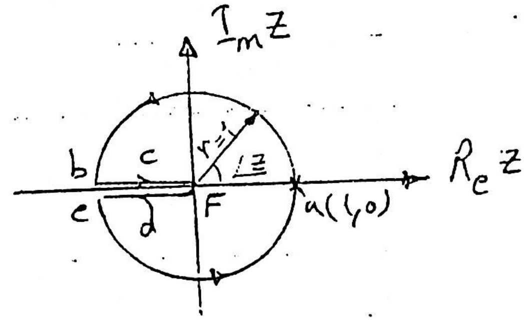
$$= \sum_{m=0}^{\infty} f_1(m) \left\{\underbrace{\sum_{K=0}^{\infty} f_2(K-m)\, z^{-K}}_{\text{shifting}}\right\}$$

$$= \sum_{m=0}^{\infty} f_1(m)\, z^{-m}\, F_2(z) = F_1(z) \cdot F_2(z)$$

Mapping of S plane into Z plane



S-plane



Z-plane (unit circle)

$$z = e^{ST} = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T} = |z| \angle z = r \angle \theta$$

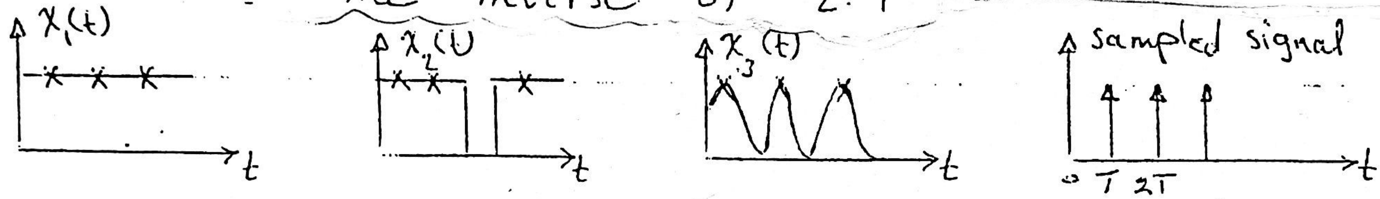
S-plane	Z-plane
$a(\overset{\sigma}{0}, \overset{\omega}{0})$	$1 \angle 0^\circ$
moving along $j\omega$ axis is	$1 \angle 0 < \theta < \pi$
$b(0, \pi)$	$1 \angle \pi$
$c(-\infty, \pi)$	$0 \angle \pi$
$F(-\infty, 0)$	$0 \angle 0$
$d(-\infty, -\pi)$	$0 \angle -\pi$
$e(0, -\pi)$	$1 \angle -\pi$

Then return to point a.

Z-

S-plane	maps into →	Z-plane
left half s-plane		inside unit circle.
Right " "		outside " "
Imaginary axis		Circumference of " "

The inverse of Z.T



The inverse of Z.T is generally not unique.

Methods of obtaining the inverse of Z.T;

1. Partial fraction method طريقة الكسور الجزئية
2. Power series inversion method
3. Discrete convolution (graphical & Tabular methods)
   جدول

1. Partial Fraction method

$$X(z) = \frac{N(z)}{D(z)}$$

$$\frac{X(z)}{z} = \frac{A_1}{z-l_1} + \frac{A_2}{z-l_2} + \cdots$$

$A_1, A_2, \ldots$ ثوابت

$l_1, l_2, \ldots$ دوال

بعد ايجاد الثوابت

$$X(z) = \frac{z A_1}{z-l_1} + \frac{z A_2}{z-l_2} + \cdots$$

$$x(n) = \mathcal{Z}^{-1}\{X(z)\}$$

Ex use Partial Fraction method to find $x(n)$ if

$$X(z) = \frac{0.632\,z}{z^2 - 1.368\,z + 0.368}$$

$$\frac{X(z)}{z} = \frac{0.632}{(z-1)(z-0.368)} = \frac{A}{z-1} + \frac{B}{z-0.368}$$

$$A = \lim_{z \to 1} \frac{0.632}{(z-0.368)} = 1$$

$$B = \lim_{z \to 0.368} \frac{0.632}{(z-1)} = -1$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-0.368}$$

$\therefore\ x(n) = u(n) - (0.368)^n\, u(n)$ أفضل طريقة لأن الدوال معروفة كـ

2. Power series inversion method أسوأ طريقة, لنفس المثال السابق نتبع عملية القسمة الطويلة

$$X(z) = 0.632\,z^{-1} + 0.86\,z^{-2} + \cdots$$

$$x(n) = 0.632\,\delta(n-1) + 0.86\,\delta(n-2) + \cdots$$

3. Discrete Convolution :-

i) Graphical method

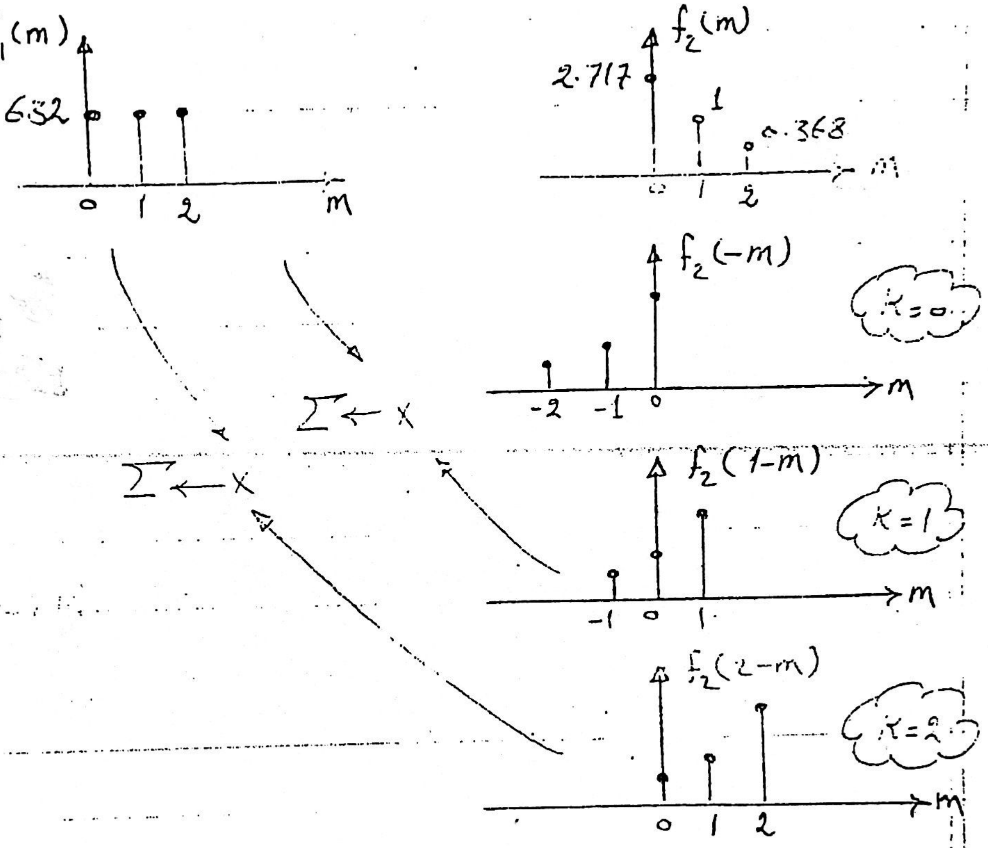
$$X(z) = \frac{0.632\,z}{(z-1)(z-0.368)} = \frac{0.632\,z}{z-1} \cdot \frac{1}{z-0.368} = X_1(z) \cdot X_2(z)$$

$$\boxed{X_1(z)} = \frac{0.632\,z}{z-1} \qquad \therefore\ \boxed{x_1(n)} = 0.632\,u(n)$$

$$X_2(z) = \frac{1}{z-0.368}\cdot\frac{z}{z} \sim\!\to x_2(n) = (0.368)^{n-1}$$

To find $f_1(n) \circledast f_2(n)$ for $n = 0, 1, 2$ and $k = 1$ & 2

تم تسمية المحاور مباشرة



$k=0$

$(2.717)(0.632) = 1.717$

$K=1$

$$\sum_{m=0}^{1} f_1(m)\, f_2(1-m)$$

$$= f_1(0) f_2(1) + f_1(1) f_2(0)$$

$$= 0.632\,(2.717) + 0.632\,(1) = 2.349$$

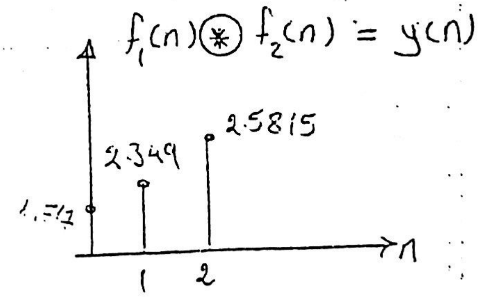
$K=2$

$$\sum_{m=0}^{2} f_1(m)\, f_2(2-m)$$

$$= f_1(0) f_2(2) + f_1(1) f_2(1) + f_1(2) f_2(0)$$

$$= 0.632\,(2.717) + 0.632\,(1) + 0.632\,(0.368)$$

$$= 2.5815$$



ii) Tabular method

$x_1(n) = 0.632$ for $n = 0, 1, 2$

$x_2(n) = 2.717,\ 1,\ 0.368$ for $n = 0, 1, 2$ respectively

		$x_1(0)$ 0.632	$x_1(1)$ 0.632	$x_1(2)$ 0.632
$x_2(0)$	2.717	1.717	1.717	1.717
$x_2(1)$	1	0.632	0.632	0.632
$x_2(2)$	0.368	0.2325	0.2325	0.2325

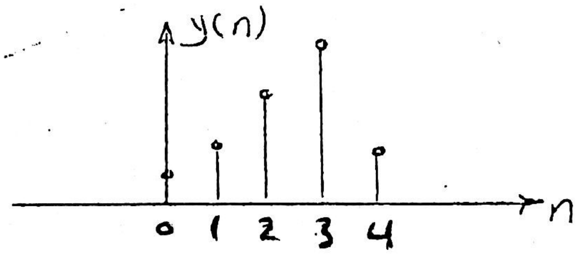
$y(0) = 1.717$

$y(1) = 1.717 + 0.632 = 2.349$

$y(2) = 2.5815$

$y(3) = 0.8671$

$y(4) = 0.2325$



Application of Z.T to Difference Equation

differential eq. $\xrightarrow[\text{signals}]{\text{continuous}}$ $y'' + 3y' + 2 = x(t)$

difference eq. $\xrightarrow[\text{signals}]{\text{discrete}}$ $y(n) + 4y(n-1) = x(n)$

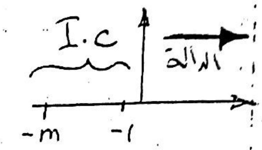
The difference eq. is given by;

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$

$$\sum_{k=0}^{N} a_k \, y(n-k) = \sum_{k=0}^{M} b_k \, x(n-k)$$

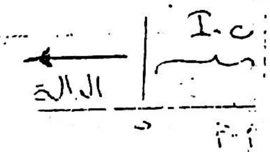
The order is N.

$$\mathcal{Z}\{x(n-m)\} = z^{-m}\left\{X(z) + \sum_{k=-m}^{-1} x(k)\,z^{-k}\right\}$$



Ex $\mathcal{Z}\{x(n-3)\} = z^{-3}\left[X(z) + x(-3)z^{3} + x(-2)z^{2} + x(-1)z\right]$

$$\mathcal{Z}\{x(n+p)\} = z^{p}\left\{X(z) - \sum_{k=0}^{p-1} x(k)\,z^{-k}\right\}$$



Ex $\mathcal{Z}\{x(n+3)\} = z^{3}\left[X(z) - x(0)z^{0} - x(1)z^{-1} - x(2)z^{-2}\right]$

e.g.:

$$\mathcal{Z}\{x(n-1)\} = z^{-1}\left\{X(z) + \sum_{k=-1}^{-1} x(k)\,z^{-k}\right\} = z^{-1}X(z) + x(-1)$$

$$\mathcal{Z}\{x(n-2)\} = z^{-2}\left\{X(z) + \sum_{k=-2}^{-1} x(k)\,z^{-k}\right\}$$

$$= z^{-2}\left\{X(z) + x(-2)z^{2} + x(-1)z\right\} = z^{-2}X(z) + x(-2) + x(-1)z^{-1}$$

Ex 1: Show $\mathcal{Z}\{x(n+1)\} = z\{X(z) - x(0)\}$

$$\mathcal{Z}\{x(n+1)\} = \sum_{n=0}^{\infty} x(n+1)\,z^{-n} = z\sum_{n=0}^{\infty} x(n+1)\,z^{-(n+1)}$$

let $k = n+1$, $n=0$, $k=1$

بالإضافة والطرح

$$= z\left\{\sum_{k=1}^{\infty} x(k)\,z^{-k} + x(0) - x(0)\right\}$$

$$= z\left\{\sum_{k=0}^{\infty} x(k)\,z^{-k} - x(0)\right\}$$

$$= z\{X(z) - x(0)\}$$

Ex.2 : Find $h(n) = y(n)/x(n)$ for $y(n) = 0.8\, y(n-1) + x(n)$

$$Y(z) = 0.8\, z^{-1}\, Y(z) + X(z)$$

$$\{1 - 0.8\, z^{-1}\}\, Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.8 z^{-1}} = \frac{z}{z - 0.8}$$

$$\therefore h(n) = (0.8)^n\, u(n)$$

Ex.3 : Solve $x(n+1) - x(n) = 0$, $x(0) = 1$

$$z\,\{X(z) - x(0)\} - X(z) = 0$$

$$\{z - 1\}\, X(z) = z$$

$$X(z) = \frac{z}{z-1}$$

$$\therefore x(n) = u(n)$$

Ex.4 : Realize the following difference eq. given below and find $h(n)$

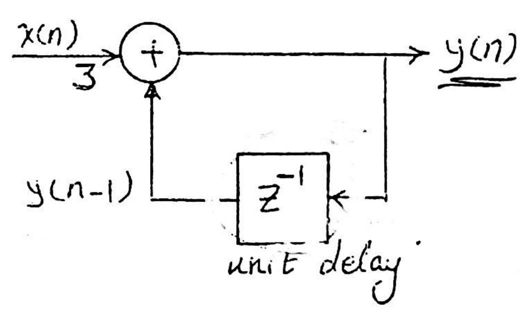
$$\boxed{y(n) = y(n-1) + 3\, x(n)}$$

$$Y(z) = z^{-1}\, Y(z) + 3\, X(z)$$

$$\{1 - z^{-1}\}\, Y(z) = 3\, X(z)$$

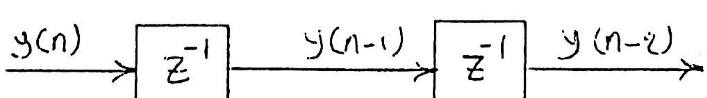
$$H(z) = \frac{Y(z)}{X(z)} = \frac{3}{1 - z^{-1}} = \frac{3z}{z-1}$$

$$\therefore h(n) = 3\, u(n)$$



$y(n) = 3\,x(n) + y(n-1)$

Note ;



Homeworks

1

The Z - Transform

1. Show that $\mathcal{Z}\{x(n-m)\} = z^{-m} X(z)$ assume zero I.C

2. Show that $\mathcal{Z}\left\{\sum_{m=0}^{k} f_1(m)\, f_2(k-m)\right\} =$

$$\mathcal{Z}\left\{\sum_{m=0}^{k} f_1(k-m)\, f_2(m)\right\} = F_1(z)\,.\,F_2(z)$$

3. Find Z.T of the following :

   a) $\mathcal{Z}\{\sin wn\}$

   b) $\mathcal{Z}\{\cos wn\}$

   c) $\mathcal{Z}\{\cosh wn\}$

   d) $\mathcal{Z}\{e^{-an} \sin wn\}$

   Ans :

   a) $z \sin \omega / (z^2 - 2z\cos\omega + 1)$

   b) $z^2 - z\cos\omega / (z^2 - 2z\cos\omega + 1)$

   c) $z^2 - z\cosh\omega / (z^2 - 2z\cosh\omega + 1)$

   d) $z e^a \sin\omega / (z^2 e^{2a} - 2z e^a \cos\omega + 1)$

4. Find $x(n)$ using partial fraction method :

   a) $X(z) = \dfrac{z}{3z^2 - 4z + 1}$

   b) $X(z) = \dfrac{z+1}{3(z-1)(z-\frac{1}{3})}$

   c) $X(z) = \dfrac{10}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$

5. Use discrete convolution to solve

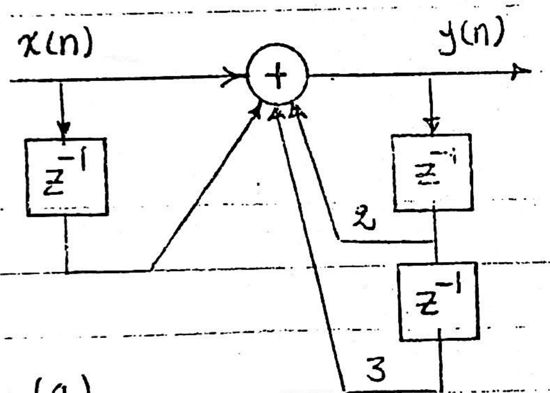
$$Y(z) = \frac{z^2}{(z-1)(z-e^{-1})} \quad , \quad n = 0, 1, 2, 3 \text{ and } K = 1, 2, 5$$

Plot $y(n)$ vs. n using Graphical method & Tabular method.

6. Write the difference equation and realize the system

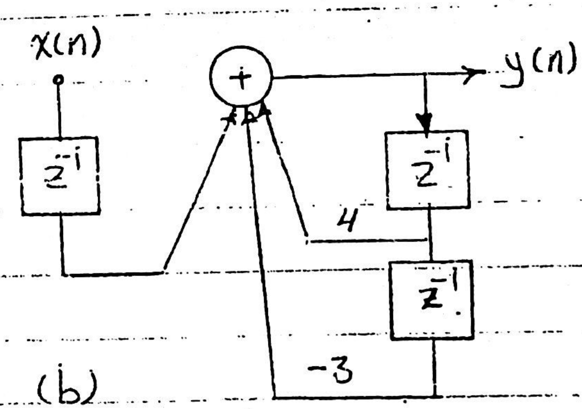
$$\frac{Y(z)}{X(z)} = \frac{8(1 - z^{-1} + z^{-2})}{2 + 5z^{-1} + 2z^{-2}}$$

7. i) Write the difference equation for the systems shown below.
ii) Find $y(n)$ / $x(n)$.



(a)

Ans: $h(n) = (3)^n u(n)$



(b)

Ans: $h(n) = \frac{1}{2}(3)^n \cancel{u(n)} - \frac{1}{2} u(n)$

8. Solve

$$y(n) - \frac{3}{2} y(n-1) + \frac{1}{2} y(n-2) = \left(\frac{1}{4}\right)^n$$

$$y(-1) = 4 \quad , \quad y(-2) = 10$$

Ans: $\frac{1}{3}\left(\frac{1}{4}\right)^n u(n) + \left(\frac{1}{2}\right)^n u(n) + \frac{2}{3} u(n)$

[illegible]

9. Solve

$$y(n) - \frac{5}{6}\, y(n-1) + \frac{1}{6}\, y(n-2) = 5^{-n}$$

$$y(-1) = 6 \quad , \quad y(-2) = 25$$

Ans; $\left\{ 5^{-n} + \frac{3}{2}\,(2)^{-n} - \frac{2}{3}\,(3)^{-n} \right\} u(n)$

10. Solve

$$x(n+2) - x(n+1) + 0.5\, x(n) = 1$$

$$x(0) = 2 \quad , \quad x(1) = 3$$

Ans; $2 + 2\left(\frac{1}{\sqrt{2}}\right)^{n} \sin\frac{n\pi}{4}$