Special Functions

1) Gamma Function

$$\Gamma(\alpha+1) = \int_0^\infty e^{-t}\, t^{x}\, dt$$

$$\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt \qquad , x > 0$$

let $u = e^{-t} \Rightarrow du = -e^{-t}\, dt$

$dv = t^{\alpha-1}\, dt \Rightarrow v = \frac{1}{x} t^{x}$

$$\therefore \Gamma(x) = \cancel{\left[e^{-t} \cdot \frac{1}{x} t^{x} \right]_0^\infty} - \int_0^\infty \frac{1}{x} t^{\alpha} \cdot -e^{-t}\, dt$$

$$= \frac{1}{x} \int_0^\infty e^{-t}\, t^{x} = \Gamma(x+1)$$

$$\therefore \Gamma(x) = \frac{1}{x}\Gamma(\alpha+1) \quad \text{or} \quad \boxed{\Gamma(\alpha+1) = x\,\Gamma(\alpha)}$$

$x = 1 \Rightarrow \Gamma(2) = 1\,\Gamma(1)$

$x = 2 \Rightarrow \Gamma(3) = 2\,\Gamma(2) = 2 \cdot 1 \cdot \Gamma(1)$

$x = 3 \Rightarrow \Gamma(4) = 3\,\Gamma(3) = 3 \cdot 2 \cdot 1\, \Gamma(1) = 3!\, \Gamma(1)$

$\therefore \boxed{\Gamma(x+1) = x!}$ $\qquad \Gamma(1) = 0! = 1$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-t}\, t^{-\frac{1}{2}}\, dt$$

let $t = x^2 \Rightarrow dt = 2x\, dx$ $\qquad$ when $t=0$ $x=0$

$t=\infty$ $x=\infty$

$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x^2} (x^2)^{-\frac{1}{2}} 2x\,dx$$

$$= 2\int_0^\infty e^{-x^2}\,dx = 2I$$

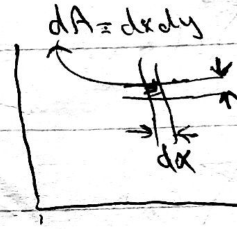
$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$dx\,dy = r\,dr\,d\theta$

$$\text{let } I = \int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-y^2}\,dy$$



$$\therefore I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

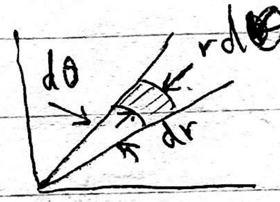
$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

$x=0$

$r=0$

$y=0$

$\theta = \frac{\pi}{2}$



$$= \left[\theta\right]_0^{\frac{\pi}{2}} \cdot -\frac{1}{2}\left[e^{-r^2}\right]_0^\infty$$

$$= \frac{\pi}{2}\cdot -\frac{1}{2}\left[0-1\right] = \frac{\pi}{4}$$

$$\therefore I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \cancel{2}\cdot\frac{\sqrt{\pi}}{\cancel{2}} = \sqrt{\pi} \qquad \therefore \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{7}{2}\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \tfrac{5}{2}\cdot\tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi}$$

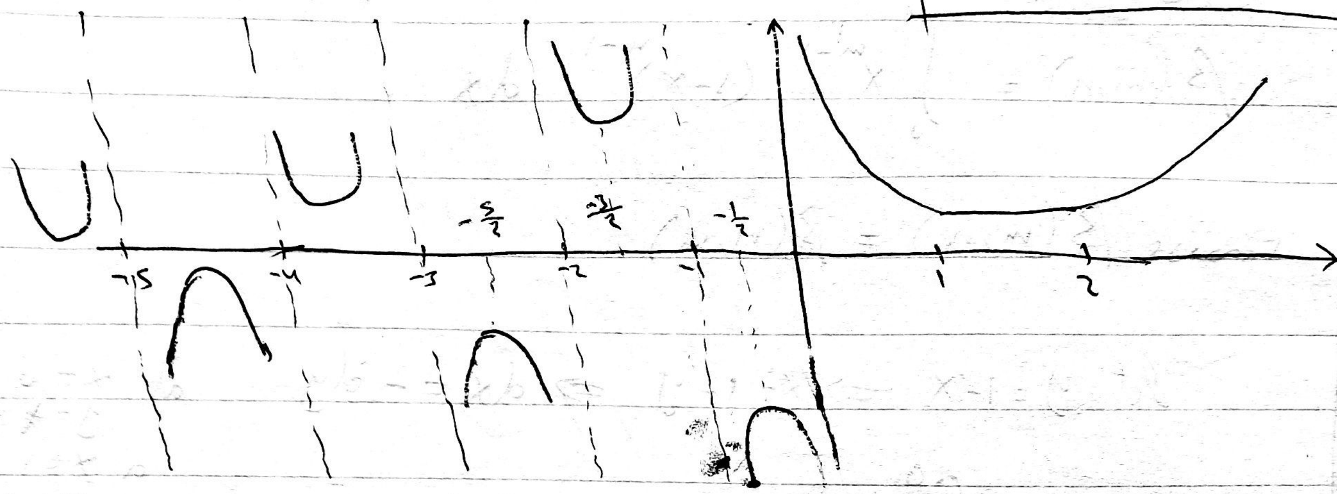
$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

$$\Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = \frac{\sqrt{\pi}}{-\frac{1}{2}} = -2\sqrt{\pi}$$

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{-2\sqrt{\pi}}{-\frac{3}{2}} = \frac{4\sqrt{\pi}}{3}$$

drawing of gamma function

$$\boxed{\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}}$$



Example find $I = \int_0^\infty e^{-y^3} \sqrt{y}\, dy$

Solution let $t = y^3 \Rightarrow dt = 3y^2\, dy \Rightarrow dy = \frac{1}{3} y^{-2}\, dt$

then $y = t^{\frac{1}{3}}$ & $y^2 = t^{\frac{2}{3}}$ & $\sqrt{y} = t^{\frac{1}{6}}$

$$\therefore I = \int_0^\infty e^{-t} . t^{\frac{1}{6}} . \frac{1}{3} t^{-\frac{2}{3}}\, dt$$

$$= \frac{1}{3} \int_0^\infty e^{-t}\, t^{\frac{1-4}{6}}\, dt = \frac{1}{3} \int_0^\infty e^{-t}\, t^{-\frac{1}{2}}\, dt$$

$$= \frac{1}{3} \Gamma\left(\frac{1}{2}\right)$$

Example find $I = \int_0^\infty 3^{-x^2}\, dx$

Solution let $t = x^2 \ln 3 \Rightarrow dt = 2x \ln 3\, dx \Rightarrow dx = \frac{dt}{2x \ln 3}$

$$x = \frac{t^{\frac{1}{2}}}{\sqrt{\ln 3}}$$

$$\therefore I = \int_0^\infty e^{-t} . \frac{1}{2 . \frac{t^{1/2}}{\sqrt{\ln 3}} \ln 3}\, dt = \frac{1}{2\sqrt{\ln 3}} \int_0^\infty e^{-t}\, t^{-\frac{1}{2}} = \frac{1}{2\sqrt{\ln 3}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{\ln 3}}$$

2) Beta Function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Prove $\beta(m,n) = \beta(n,m)$

let $y = 1-x \Rightarrow x = 1-y \Rightarrow dx = -dy$ at $x=0$, $y=1$; at $x=1$, $y=0$

$$\therefore \beta(m,n) = \int_1^0 (1-y)^{m-1}\, y^{n-1} \cdot -dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n,m)$$

let $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$ at $x=0$, $\sin^2\theta = 0$, $\theta = 0$; at $x=1$, $\sin^2\theta = 1$, $\theta = \frac{\pi}{2}$

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$\boxed{\therefore \beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta}$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\beta\left(\tfrac{1}{2},2\right) = \frac{\Gamma(\frac{1}{2})\Gamma(2)}{\Gamma(\frac{3}{2})} = \frac{\sqrt{\pi}\cdot 1!}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{4}{3}$$

Beta Function

(1) $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$

$\beta(m,n) = \beta(n,m)$

$x = \sin^2 \theta$

(2) $\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \, d\theta$

$x = \frac{y}{1+y}$ $\qquad 1-x = 1-\frac{y}{1+y} = \frac{1}{1+y}$

$dx = \frac{(1+y)-y}{(1+y)^2} \, dy$

or $dx = \frac{1}{1+y^2} \, dy$

$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow x = \frac{y}{1+y}$

$x + xy = y$

$x = y(1-x)$

$y = \frac{x}{1-x} = \infty$

$\therefore \beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{1+y}\right)^{n-1} \frac{1}{(1+y)^2} \, dy$

(3) $\therefore \beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} \, dy$

Example (1) find $I = \int_0^b x^4 \sqrt{b^2 - x^2} \, dx$

Solution $I = b\int_0^b x^4 \sqrt{1-\left(\frac{x}{b}\right)^2} \, dx$

let $y = \left(\frac{x}{b}\right)^2 \Rightarrow x = y^{\frac{1}{2}} * b \qquad dx = b\,\frac{1}{2}\,y^{-\frac{1}{2}}\,dy$

$\therefore I = b\int_0^1 b^4 y^2 (1-y)^{\frac{1}{2}}\, b \cdot \frac{1}{2} y^{-\frac{1}{2}}\, dy \qquad x = 0 \quad y = 0$

$\qquad x = b \quad y = 1$

$= \frac{b^6}{2}\int_0^1 y^{3/2} (1-y)^{\frac{1}{2}}\, dy \qquad \therefore m-1 = \frac{3}{2}$

$\qquad n-1 = \frac{1}{2}$

$= \frac{b^6}{2}\beta\left(\frac{5}{2}, \frac{3}{2}\right) \qquad m = \frac{5}{2}, \; n = \frac{3}{2}$

$$= \frac{b^6}{2}\,\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)} = \frac{b^6}{2}\,\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{3!} = \frac{b^6\pi}{32}$$

<u>Example 2</u> find $I = \int_0^{\pi} \cos^4\theta\, d\theta$

<u>Solution</u> $I = \int_0^{\pi/2} \cos^4\theta\, d\theta + \int_{\pi/2}^{\pi} \cos^4\theta\, d\theta$

let $\theta = \pi - \alpha \Rightarrow \alpha = \pi - \theta$

$d\theta = -d\alpha \qquad \theta = \frac{\pi}{2} \quad \alpha = \frac{\pi}{2}$

$\qquad \theta = \pi \quad \alpha = 0$

$= \int_0^{\pi/2} \cos^4\theta\, d\theta + \int_{\pi/2}^{0} \cos^4\alpha\,(-d\alpha) \qquad \cos\theta = \cos(\pi - \alpha)$

ربع ثاني $\quad = -\cos\alpha$

$= \int_0^{\pi/2} \cos^4\theta\, d\theta + \int_0^{\pi/2} \cos^4\alpha\, d\alpha$

مجرد رموز لا تؤثر ممكن نفرض γ أفضل
حتى نميزه عن الدالة β

$= 2\int_0^{\pi/2} \cos^4\beta\, d\beta$

$2m-1 = 0 \qquad 2n-1 = 4$

$m = \frac{1}{2} \qquad n = \frac{5}{2}$

$= \beta\left(\frac{1}{2}, \frac{5}{2}\right)$

$= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(3)} = \frac{\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2!}$

$= \frac{3}{8}\pi$

Example: Prove that $\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n\sin(\frac{m\pi}{n})}$

Solution: let $y = x^n \Rightarrow x = y^{\frac{1}{n}} \Rightarrow dx = \frac{1}{n} y^{\frac{1}{n}-1}\,dy$

$\therefore \int_0^\infty \frac{y^{\frac{m-1}{n}}}{1+y} \cdot \frac{1}{n} \cdot y^{\frac{1}{n}-1}\,dy$ $\quad x=0 \; y=0$, $x=\infty \; y=\infty$

$= \frac{1}{n}\int_0^\infty \frac{y^{\frac{m-1+1-n}{n}}}{(1+y)}\,dy$ $\quad \beta(a,b) = \int_0^\infty \frac{y^{a-1}}{(1+y)^{a+b}}\,dy$

$a = \frac{m}{n}$

$= \frac{1}{n}\int_0^\infty \frac{y^{\frac{m}{n}-1}}{(1+y)}\,dy$ $\quad a+b=1 \Rightarrow b = 1-\frac{m}{n}$

$= \frac{1}{n}\beta\left(\frac{m}{n}, 1-\frac{m}{n}\right) = \frac{1}{n}\,\frac{\Gamma(\frac{m}{n})\Gamma(1-\frac{m}{n})}{\Gamma(1)}$

$= \frac{1}{n}\cdot\frac{\pi}{\sin\frac{m\pi}{n}}$ $\quad$ where $\Gamma(\alpha)\cdot\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}$

Example: Prove that $\int_0^1 x^m (\ln(x))^n\,dx = \frac{(-1)^n\,n!}{(m+1)^{n+1}}$

where n +ve integer

Solution $\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt$

let $x = e^{-t} \Rightarrow \ln x = \ln e^{-t} = -t$

$\therefore t = -\ln x$ & $dx = -e^{-t}\, dt$

$x = 0 \quad t = \infty$

$x = 1 \quad t = 0$

قلبت بين الحدود

$$\therefore \int_\infty^0 e^{-mt} (-t)^n (-e^{-t})\, dt$$

$$= (-1)^n \int_0^\infty e^{-(m+1)t}\, t^n\, dt$$

let $(m+1)t = y \Rightarrow t = \frac{1}{m+1} y$ حدود التكامل لم تتغير

$$\therefore (-1)^n \int_0^\infty e^{-y} \left(\frac{y}{m+1}\right)^n \cdot \frac{1}{m+1} dy$$

$x-1 \Leftarrow x = n+1$ n ل

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-y}\, y^n\, dy = \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1)$$

GAMMA FUNCTION

Homeworks

1 Evaluate (a) $\int_0^\infty x^4 e^{-x}\,dx$, (b) $\int_0^\infty x^6 e^{-3x}\,dx$, (c) $\int_0^\infty x^2 e^{-2x^2}\,dx$.

Ans. (a) 24, (b) $\frac{80}{243}$, (c) $\frac{\sqrt{2\pi}}{16}$

2 Find (a) $\int_0^\infty e^{-x^3}\,dx$, (b) $\int_0^\infty \sqrt[4]{x}\,e^{-\sqrt{x}}\,dx$, (c) $\int_0^\infty y^3 e^{-2y^5}\,dy$.

Ans. (a) $\frac{1}{3}\Gamma(\frac{1}{3})$, (b) $\frac{3\sqrt{\pi}}{2}$, (c) $\frac{\Gamma(4/5)}{5\sqrt[5]{16}}$

$\frac{1}{2}\Gamma\frac{1}{2}$

3 Show that $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{s}}$, $s > 0$. ← 8 بدل

4 Prove that $\Gamma(v) = \int_0^1 \left(\ln\frac{1}{x}\right)^{v-1} dx$, $v > 0$.

5 Evaluate (a) $\int_0^1 (\ln x)^4\,dx$, (b) $\int_0^1 (x\ln x)^3\,dx$, (c) $\int_0^1 \sqrt[3]{\ln(1/x)}\,dx$.

Ans. (a) 24, (b) $-3/128$, (c) $\frac{1}{3}\Gamma(\frac{1}{3})$

6 Evaluate (a) $\Gamma(-7/2)$, (b) $\Gamma(-1/3)$. *Ans.* (a) $(16\sqrt{\pi})/105$, (b) $-3\Gamma(2/3)$

7 Prove that $\lim_{x\to -m} |\Gamma(x)| = \infty$ where $m = 0, 1, 2, 3, \ldots$

8 Prove that if m is a positive interger, $\Gamma(-m+\frac{1}{2}) = \frac{(-1)^m 2^m \sqrt{\pi}}{1\cdot 3\cdot 5\cdots(2m-1)}$

9 Evaluate (a) $\frac{\Gamma(7)}{2\,\Gamma(4)\,\Gamma(3)}$, (b) $\frac{\Gamma(3)\,\Gamma(3/2)}{\Gamma(9/2)}$, (c) $\Gamma(1/2)\,\Gamma(3/2)\,\Gamma(5/2)$.

Ans. (a) 30, (b) 16/105, (c) $\frac{3}{8}\pi^{3/2}$

BETA FUNCTION

1 Evaluate (a) $B(3,5)$, (b) $B(3/2,2)$, (c) $B(1/3,2/3)$. *Ans.* (a) 1/105, (b) 4/15, (c) $2\pi/\sqrt{3}$

2 Find (a) $\int_0^1 x^2(1-x)^3\,dx$, (b) $\int_0^1 \sqrt{(1-x)/x}\,dx$, (c) $\int_0^2 (4-x^2)^{3/2}\,dx$.

Ans. (a) 1/60, (b) $\pi/2$, (c) 3π

3 Evaluate (a) $\int_0^4 u^{3/2}(4-u)^{5/2}\,du$, (b) $\int_0^3 \frac{dx}{\sqrt{3x-x^2}}$. *Ans.* (a) 12π, (b) π

4 Prove that $\int_0^a \frac{dy}{\sqrt{a^4-y^4}} = \frac{[\Gamma(1/4)]^2}{4a\sqrt{2\pi}}$.

5 Evaluate (a) $\int_0^{\pi/2} \sin^4\theta\cos^4\theta\,d\theta$, (b) $\int_0^{2\pi} \cos^6\theta\,d\theta$. *Ans.* (a) $3\pi/256$, (b) $5\pi/8$

6 Evaluate (a) $\int_0^\pi \sin^5\theta\,d\theta$, (b) $\int_0^{\pi/2} \cos^5\theta\sin^2\theta\,d\theta$. *Ans.* (a) 16/15, (b) 8/105

7 Prove that $\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \pi/\sqrt{2}$.

8 Prove that (a) $\int_0^\infty \frac{x\,dx}{1+x^6} = \frac{\pi}{3\sqrt{3}}$, (b) $\int_0^\infty \frac{y^2\,dy}{1+y^4} = \frac{\pi}{2\sqrt{2}}$.

9 Prove that $\int_{-\infty}^\infty \frac{e^{2x}}{ae^{3x}+b}\,dx = \frac{2\pi}{3\sqrt{3}\,a^{2/3}b^{1/3}}$ where $a, b > 0$.

10 Prove that $\int_{-\infty}^\infty \frac{e^{2x}}{(e^{3x}+1)}\,dx = \frac{2\pi}{9\sqrt{3}}$